Energy transport in oscillators chains subject to a magnetic field

Gaëtan Cane

LJAD, UCA

Nice

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?

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Energy of the system

$$E(t) = \frac{1}{2} \sum_{x \in \Lambda} m_x |v(t,x)|^2 + \frac{1}{2} \sum_{x \in \Lambda} \big(q(t,x) - q(t,x-1)\big)^2 + W\big(q(t,x) - q(t,x-1)\big) = \sum_{x \in \Lambda} e(t,x)$$

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Harmonic chain: $W = 0$

Fourier's law and harmonic chain

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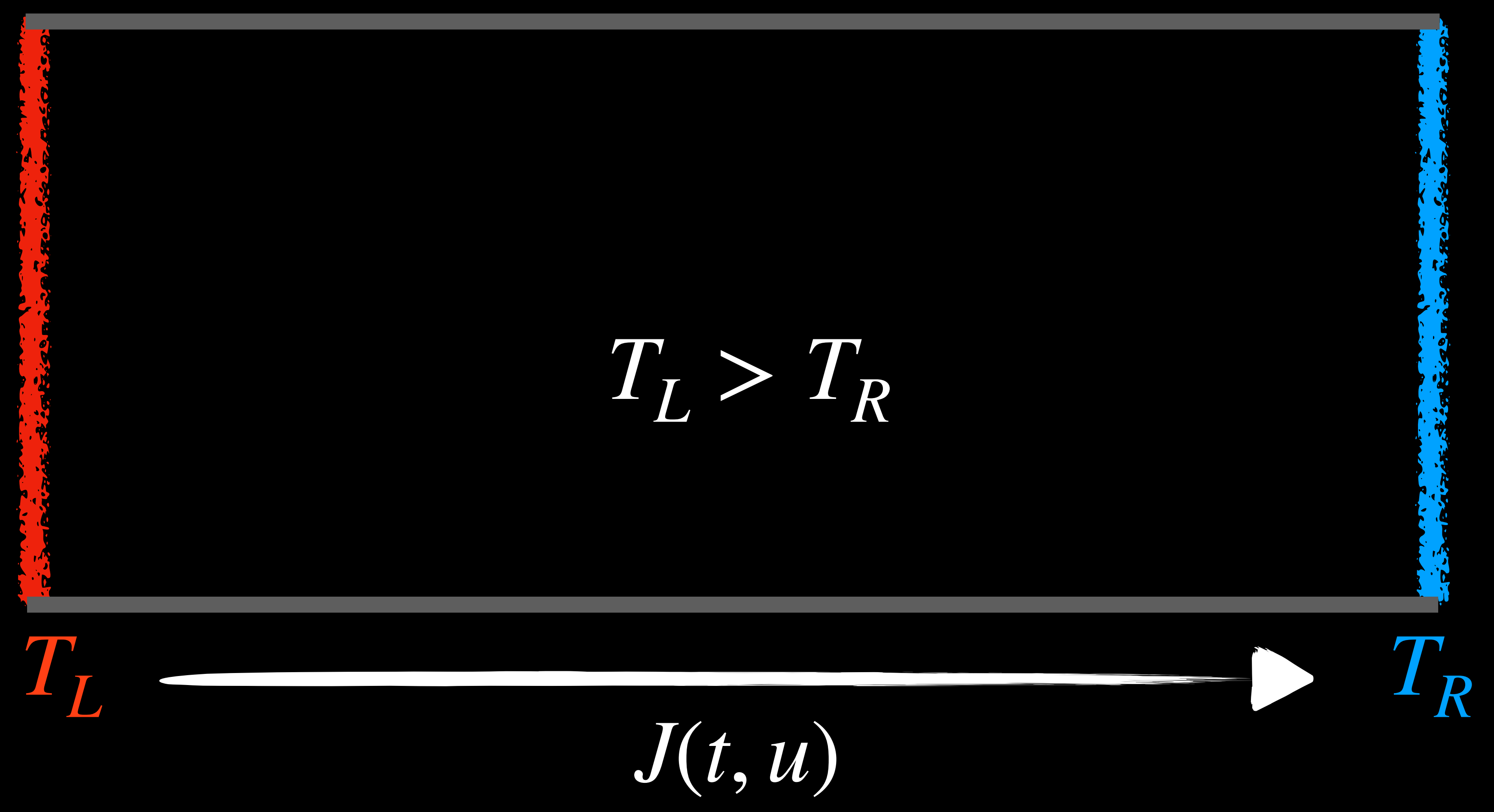
1822 : Fourier's phenomenological law



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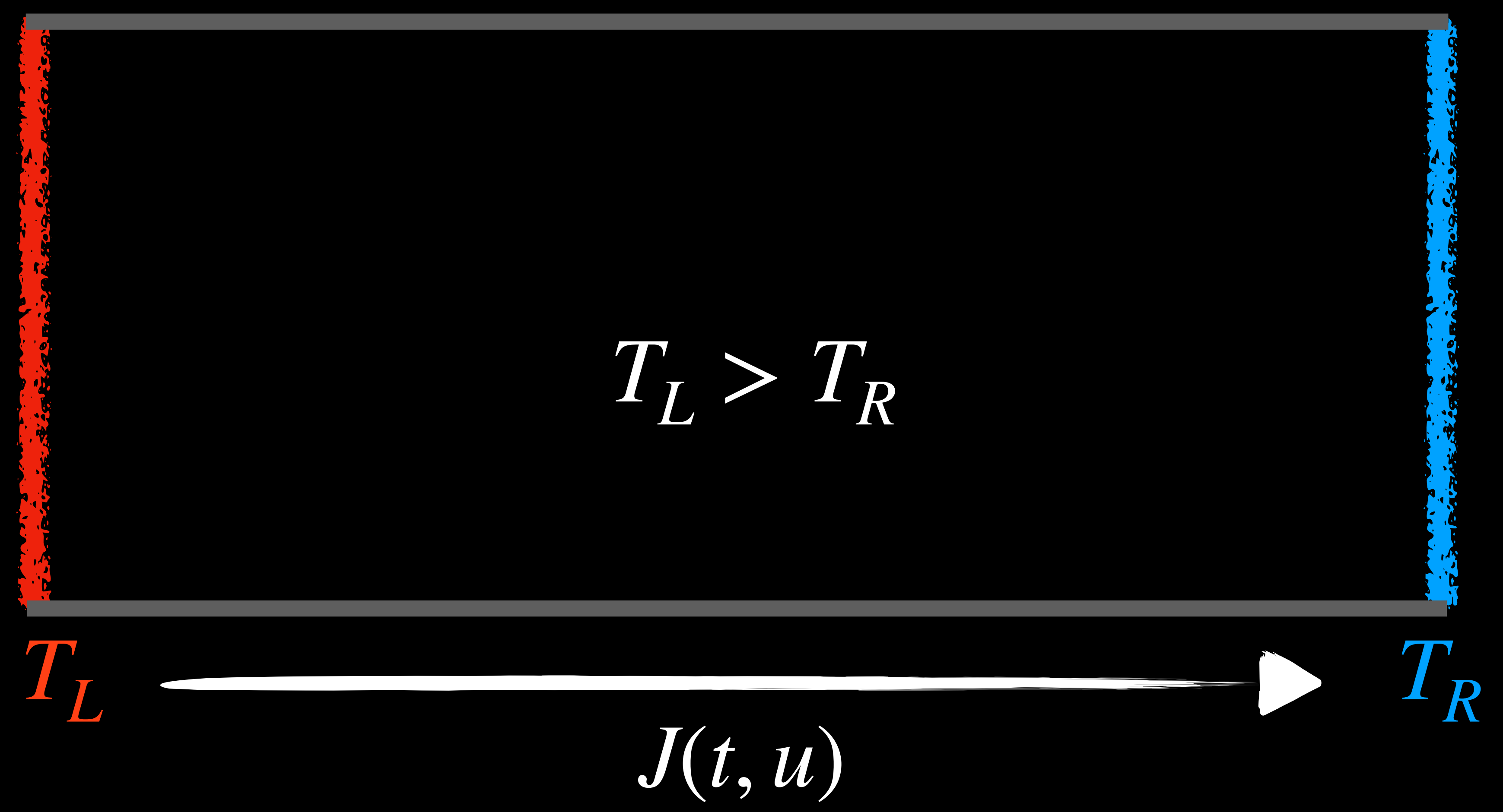

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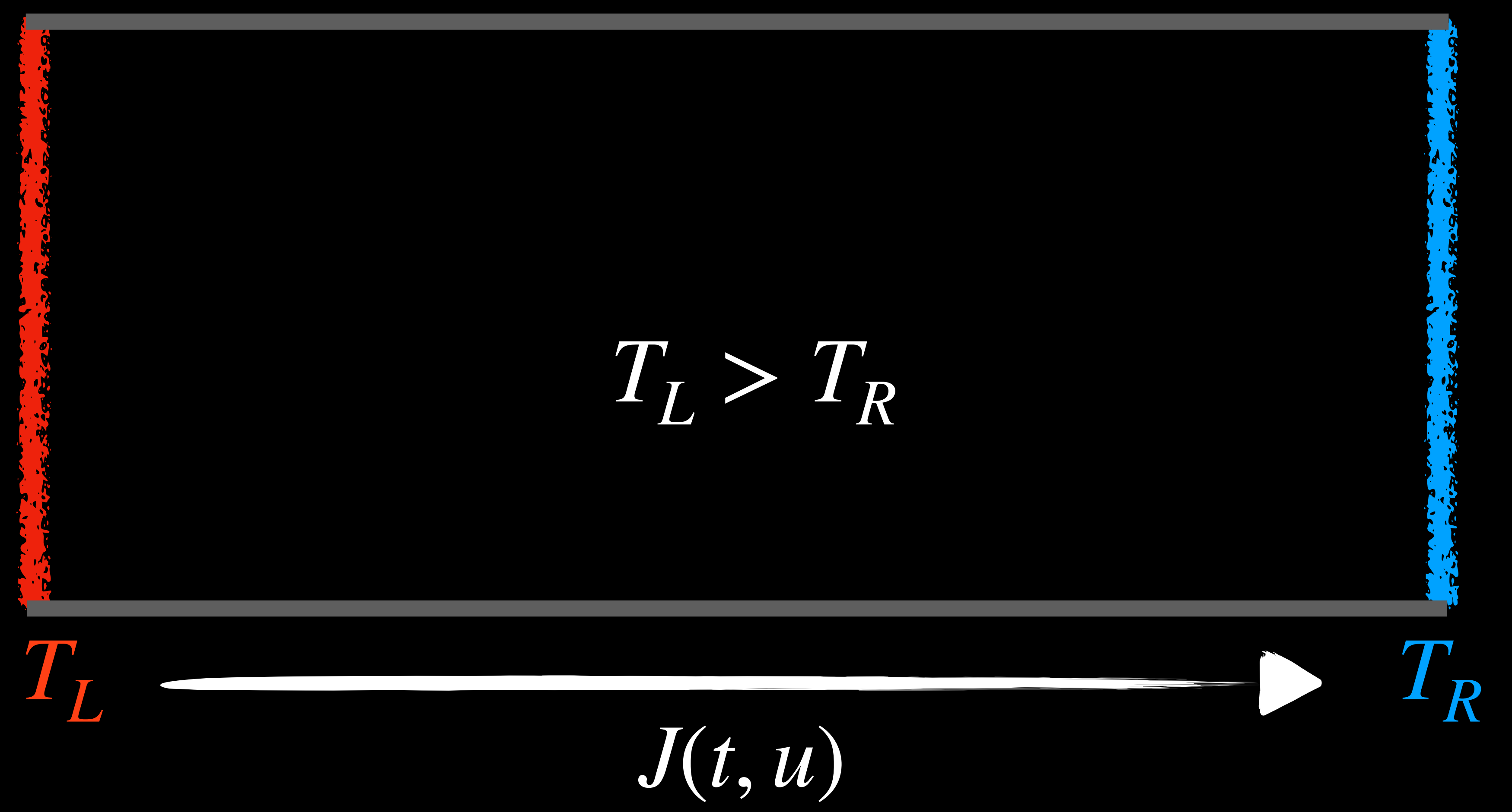

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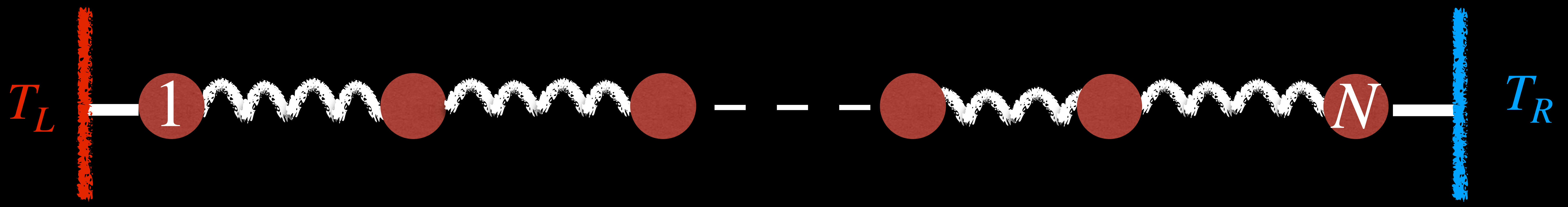

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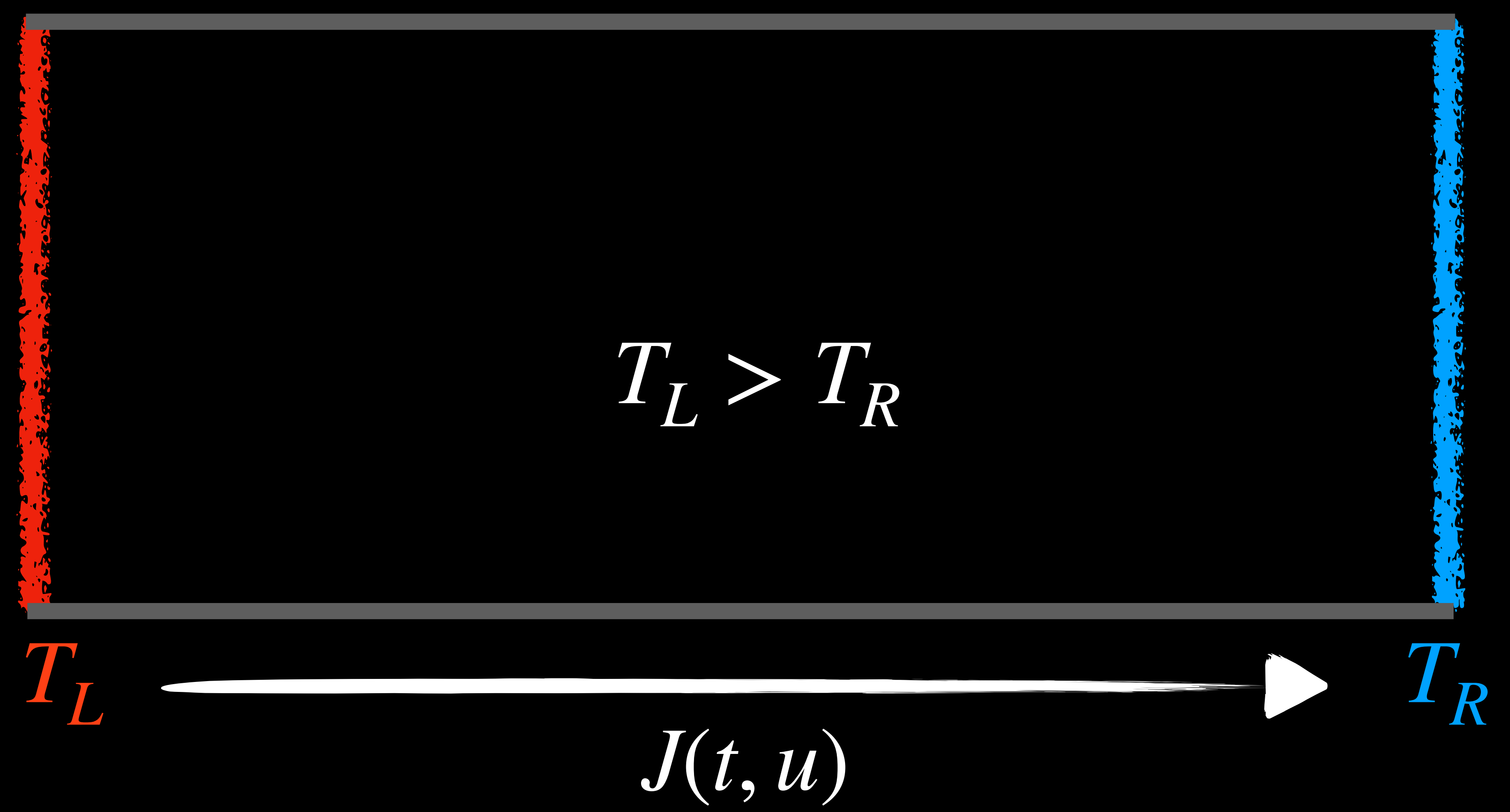

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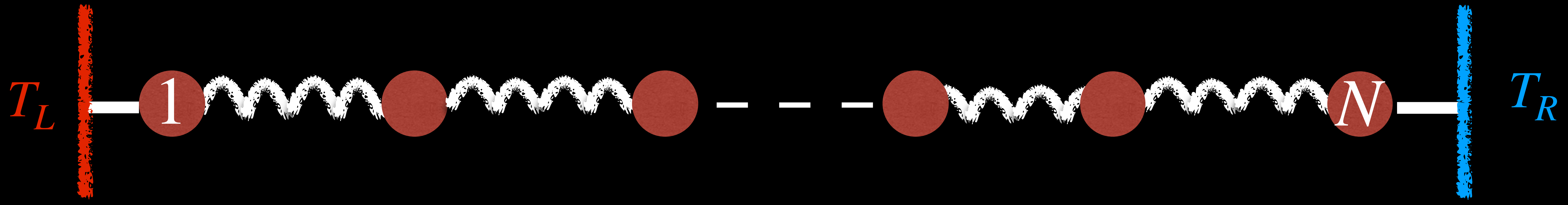

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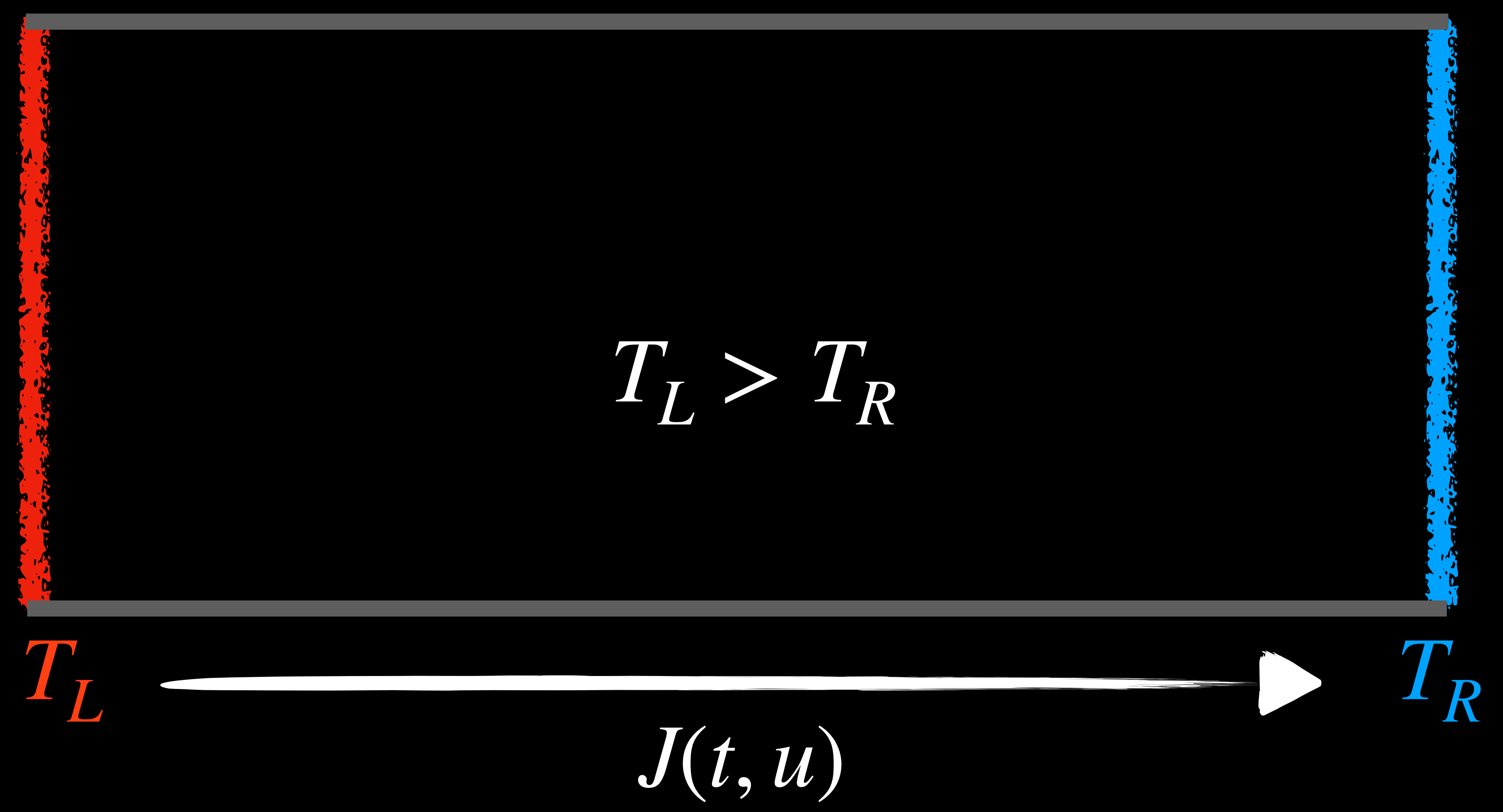

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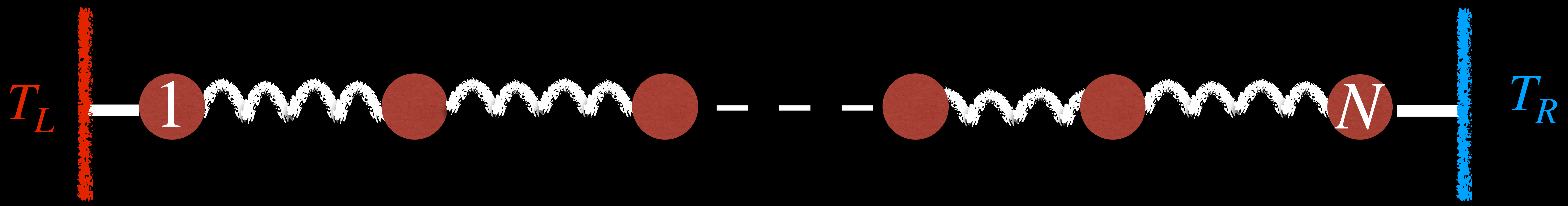

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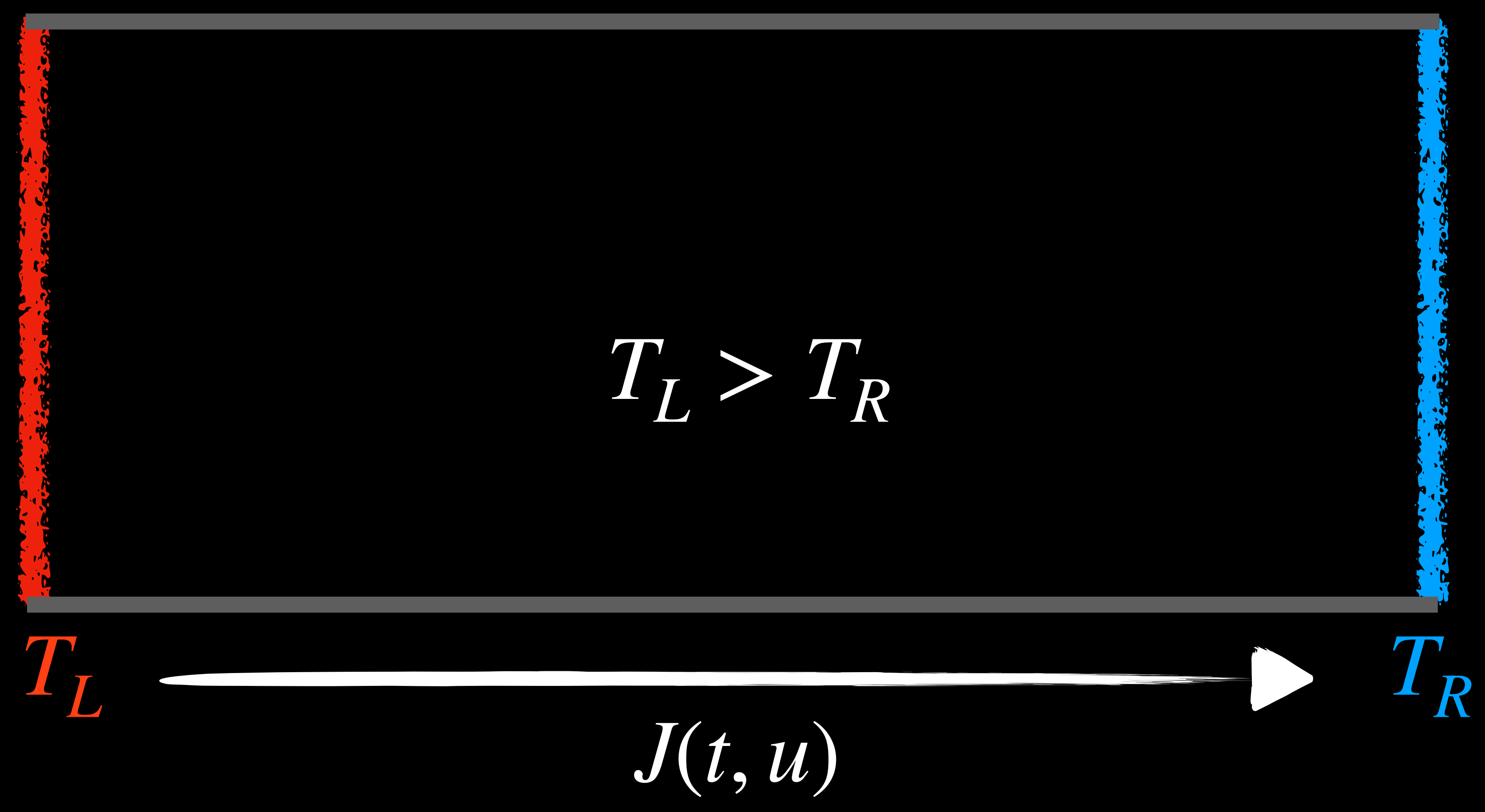

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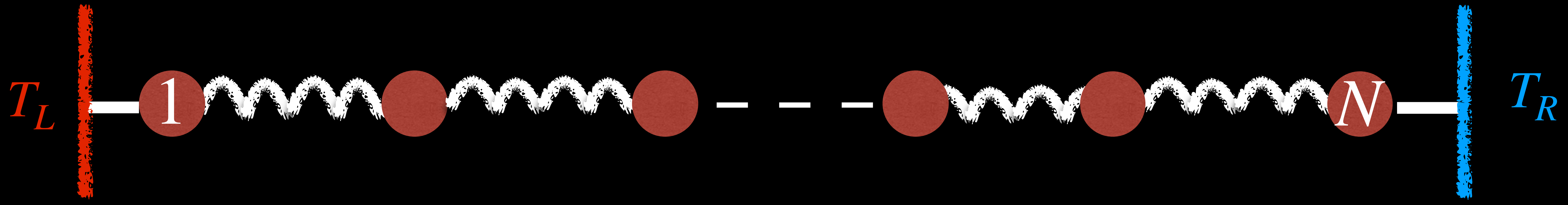

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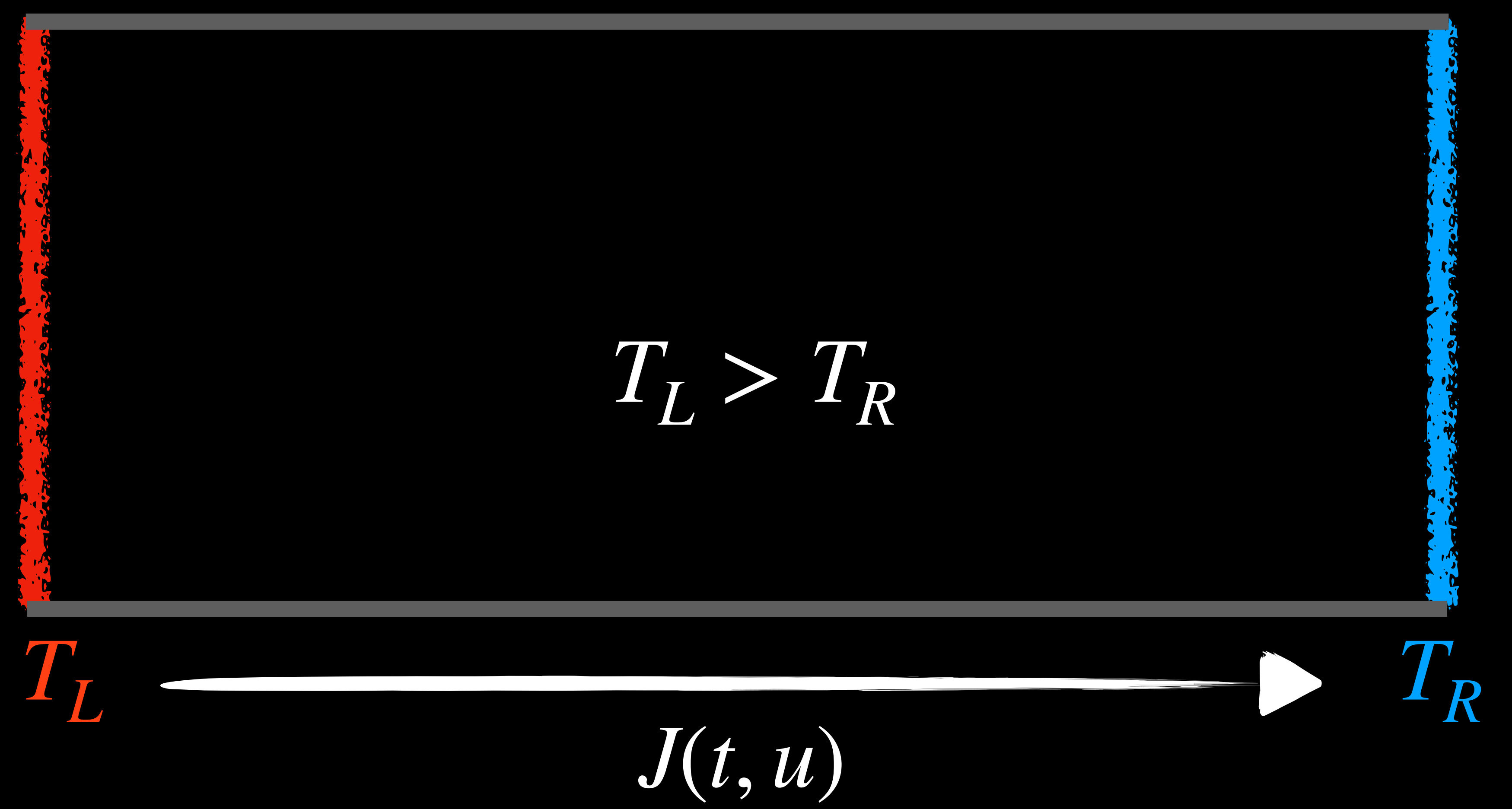

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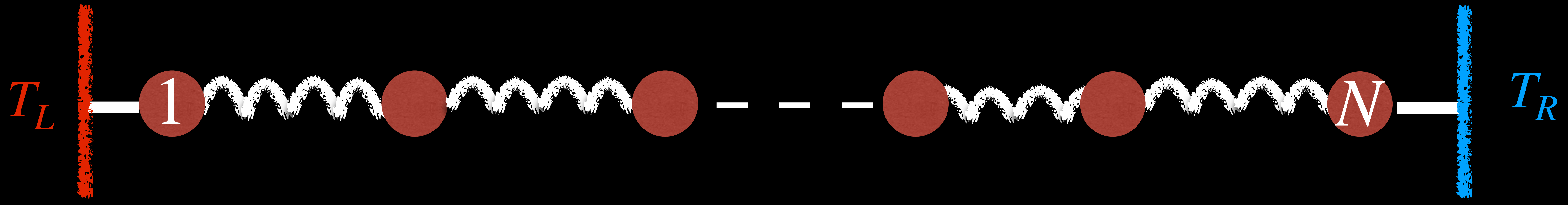

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Proved by Ajanki and Huveneers [CMP'11]

Part I
Study of the noisy harmonic chain submitted to a constant magnetic field

Noisy harmonic chain submitted to a magnetic field

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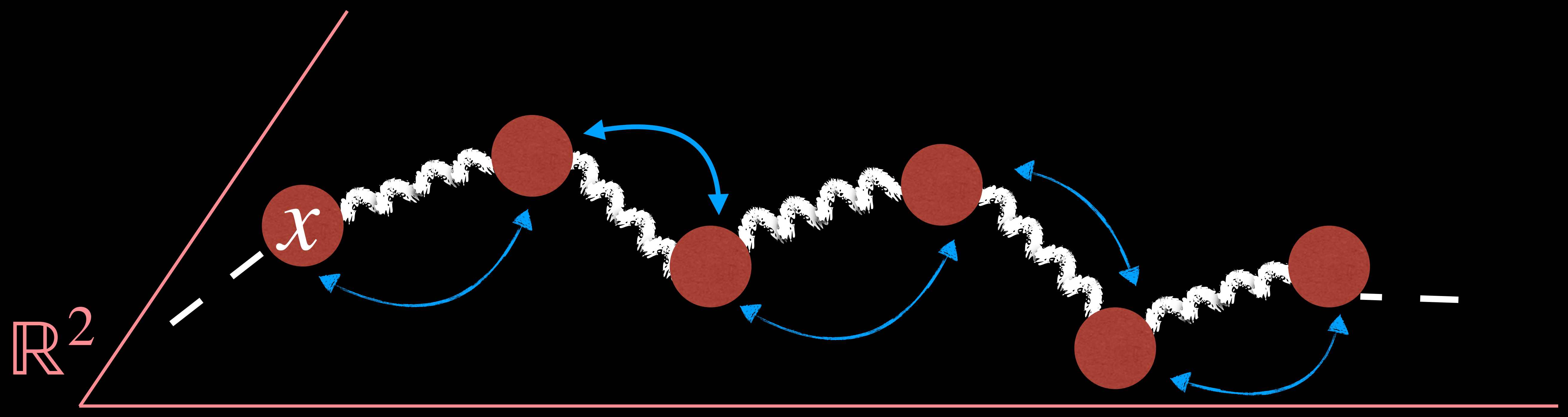

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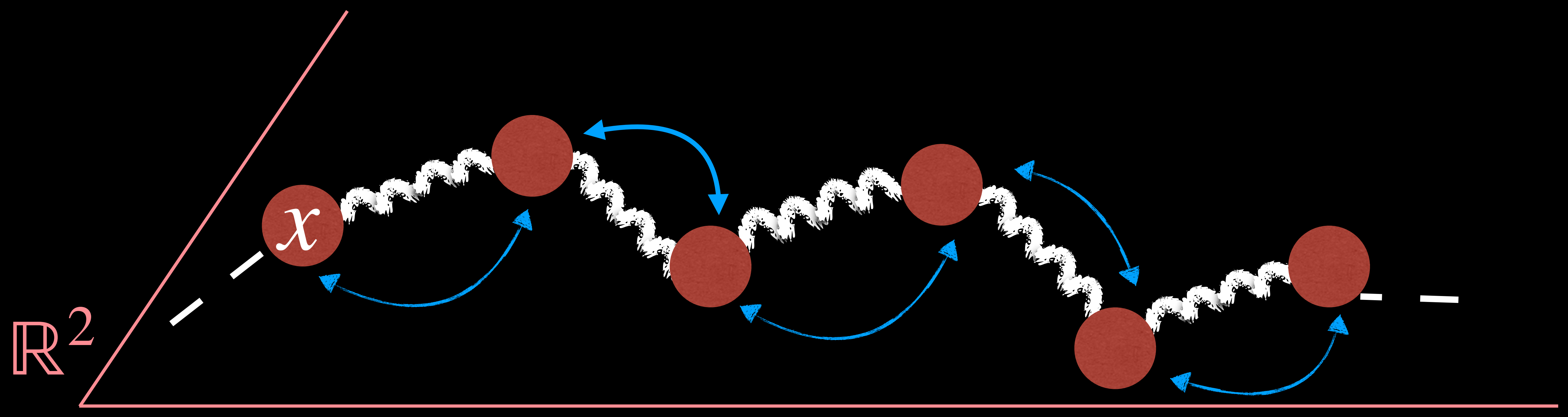

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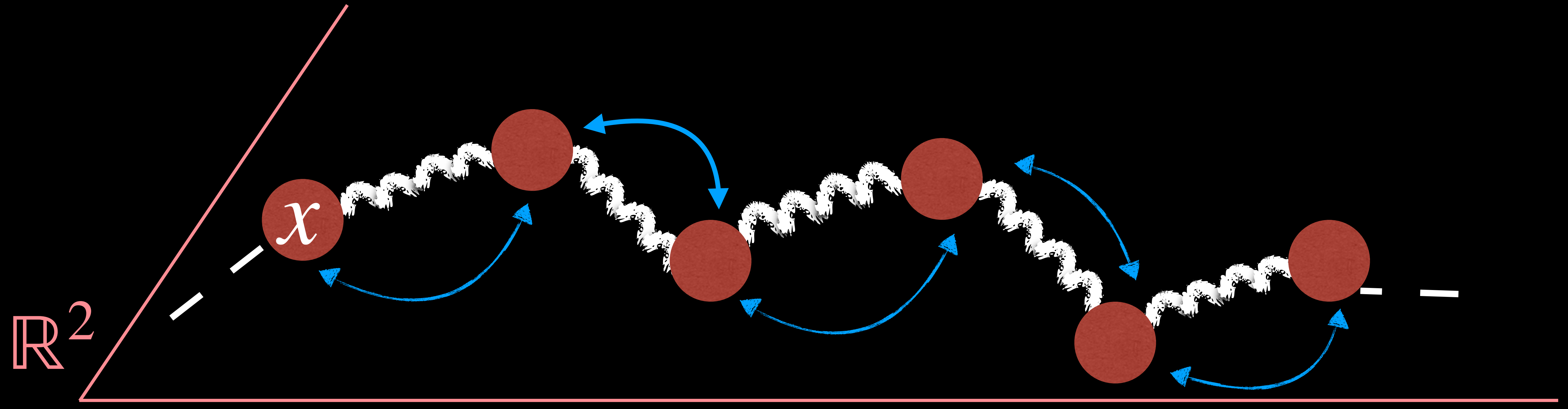

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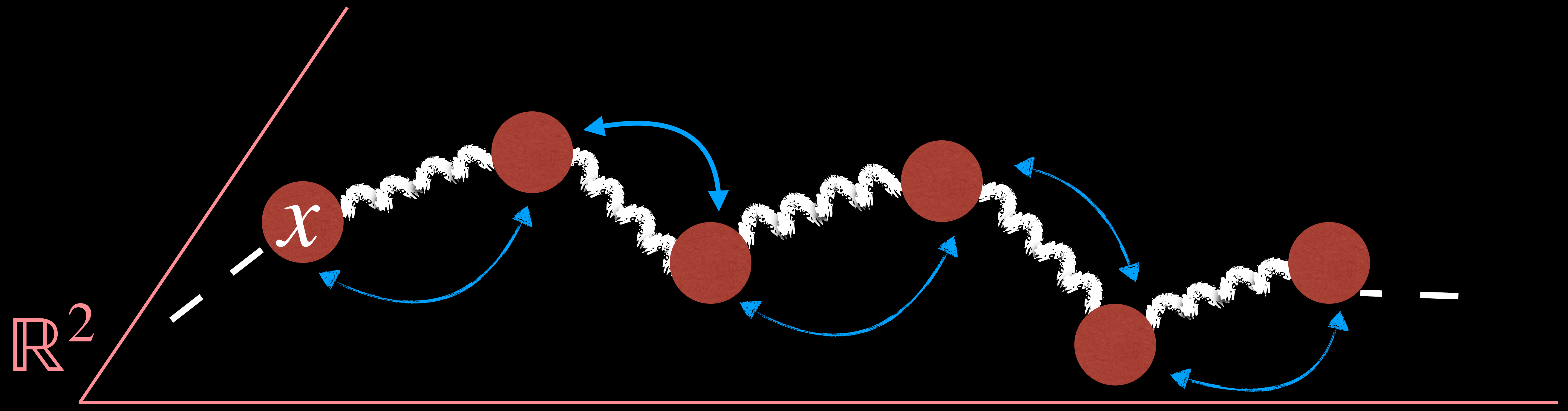

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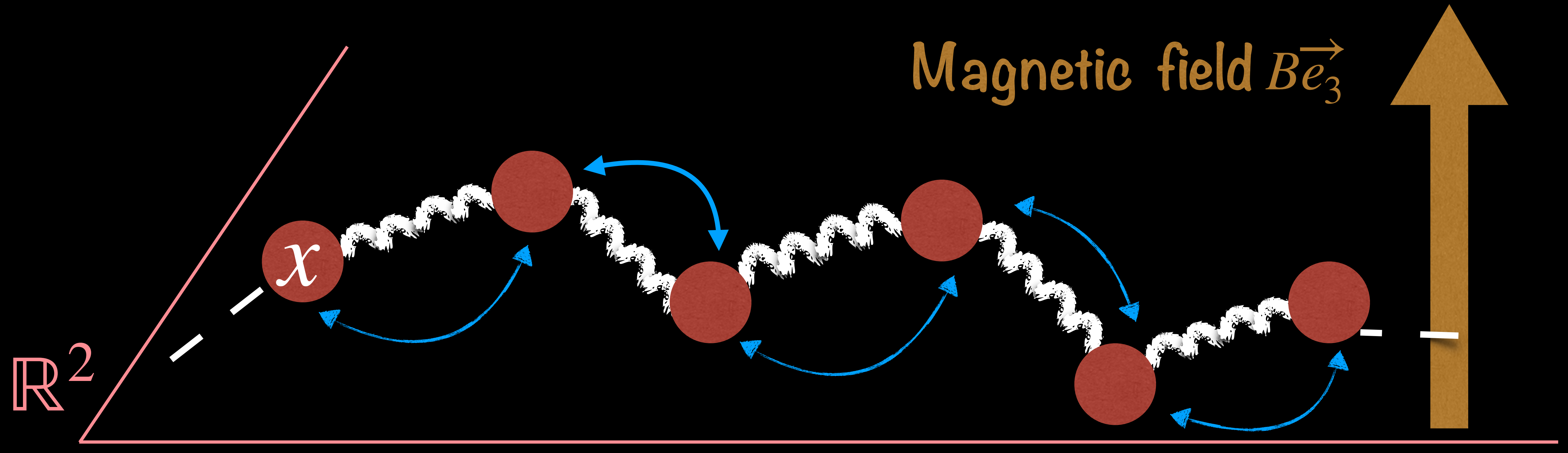

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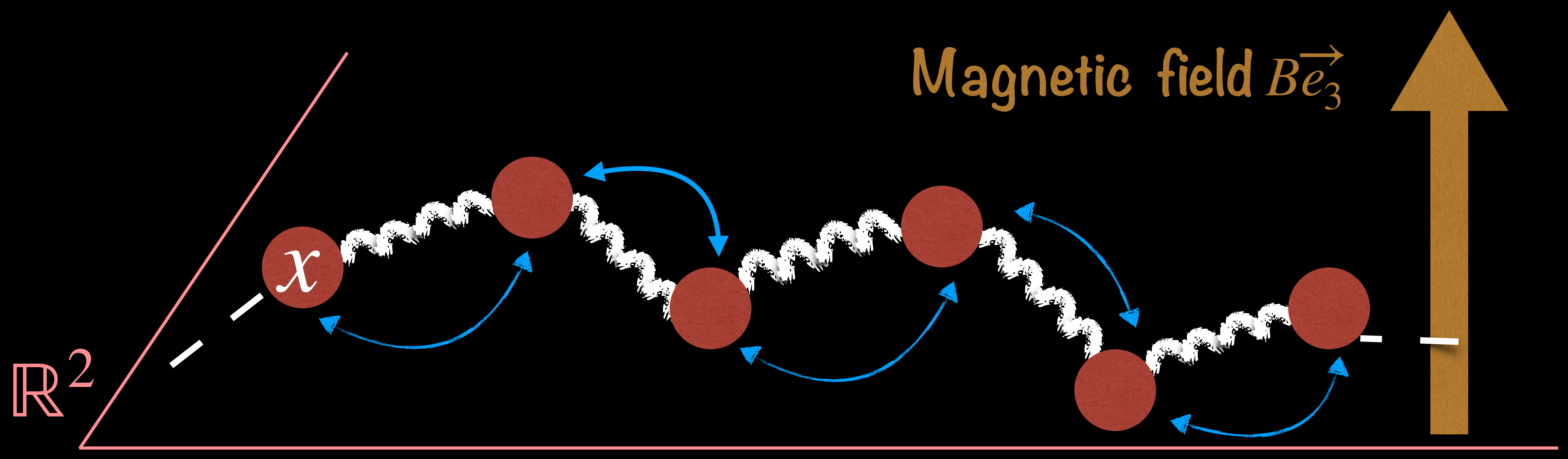

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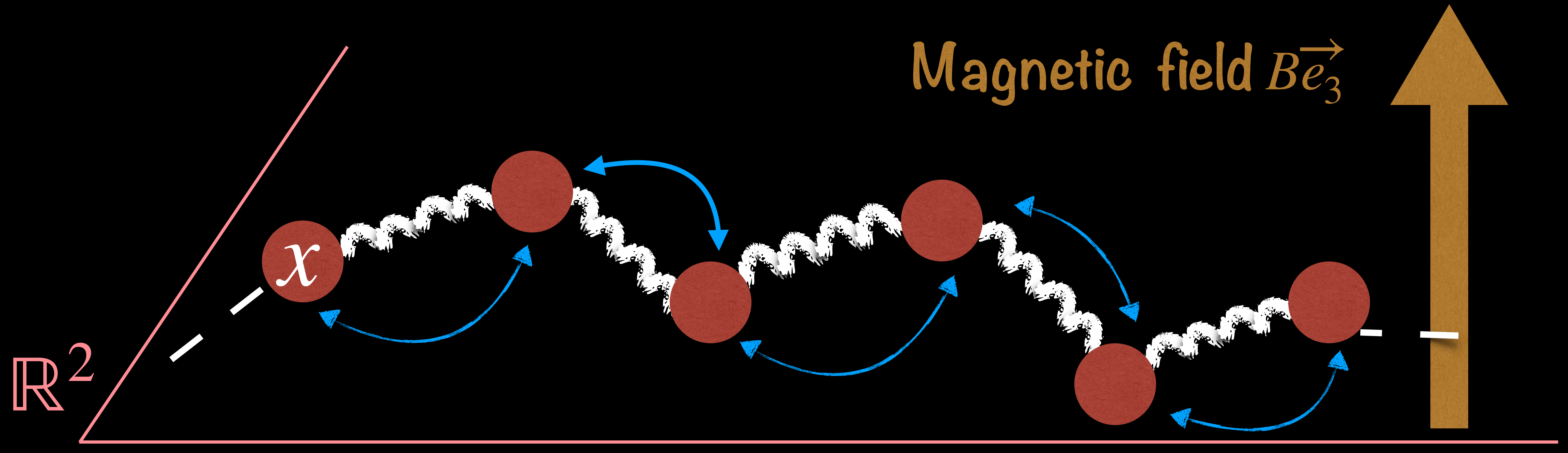

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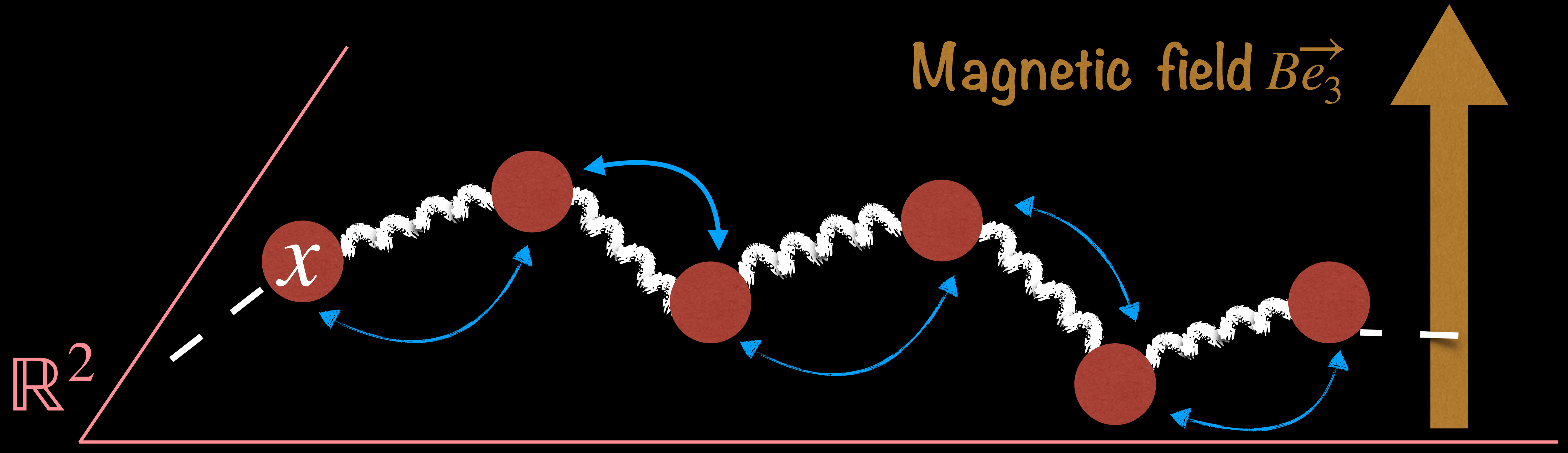

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They proved that the energy density satisfies a fractional diffusion equation with exponent $5/6$

$$\partial_t \rho_B(t,u) = -(-\Delta)^{\frac{\alpha_B}{2}} \rho_B(t,u) \quad \text{and} \quad \frac{\alpha_B}{2} = \frac{5}{6} \text{ if } B \neq 0 \text{ and } \frac{\alpha_B}{2} = \frac{3}{4} \text{ if } B = 0$$

Aim of the study

Let μ^ε be the initial distribution of the system and assume that

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}\left[e(0,x)\right] = \int_{\mathbb{R}} J(u)\mathscr{W}_0(u)du$$

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($\mathbb{T} = [0,1)$ is the unit torus)

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$$\langle \mathscr{W}^\varepsilon(t), J\rangle = \sum_{i=1}^{2}\int_{\mathbb{R}} dp \int_{\mathbb{T}} dk \mathbb{E}_{\mu_\varepsilon}\left[\hat\psi_i^*\left(t\varepsilon^{-1}, k-\frac{\varepsilon p}{2}\right)\hat\psi_i\left(t\varepsilon^{-1}, k+\frac{\varepsilon p}{2}\right)\right]\mathscr{F}\left[J_i\right](k,p)$$

Aim of the study

Let μ^ε be the initial distribution of the system and assume that

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}\left[e(0,x)\right] = \int_{\mathbb{R}} J(u)\mathscr{W}_0(u)du$$

Macroscopic scale

Can we prove that

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}\left[e\left(t\varepsilon^{-1}, x\right)\right] = \int_{\mathbb{R}} J(u)\mathscr{W}_t(u)du$$

where $\mathscr{W}_t$ solves some evolution equation?

Let $\{\hat{\psi}_{1,2}(k)\}_{k\in\mathbb{T}}$ be the eigenvectors associated to the infinitesimal generator of the deterministic dynamic.

$$\int_{\mathbb{T}} \left(\left|\hat{\psi}_1(t,k)\right|^2 + \left|\hat{\psi}_2(t,k)\right|^2 \right) dk = E(t) = E(0) \qquad \left(\mathbb{T} = [0,1) \text{ is the unit torus}\right)$$

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$$\langle \mathscr{W}^\varepsilon(t), J\rangle = \varepsilon \sum_{x\in\mathbb{Z}} \mathbb{E}_{\mu^\varepsilon}\left[e\left(t\varepsilon^{-1},x\right)\right] J(\varepsilon x) + \mathcal{O}_J(\varepsilon)$$

To understand the behavior of the energy, we have to understand the one of $\mathscr{W}^\varepsilon$.

Previous results on this model

BOS [ARMA'10] and SSS [CMP'19] proved that $\mathscr{W}^\varepsilon$ converges to f_B where

$$\partial_t f_B(t,u,k,i) + \frac{\mathbf{v}_B(k)}{2\pi}\partial_u f_B(t,u,k,i) = \mathscr{L}_B[f_B](t,u,k,i)$$

with f^0 as initial condition

$\mathscr{L}_B$ is the infinitesimal generator of a jump process $\left(K^B(\cdot), I^B(\cdot)\right) \in \left(\mathbb{T}\times\{1,2\}\right)$.

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JKO [AAP'09] (for $B=0$) and SSS [CMP'19] (for $B\neq 0$) proved that

$$\lim_{N\to\infty}\sum_{i=1}^2\int_{\mathbb{T}}\left|f_B\left(N^{\alpha_B}t, Nu, k, i\right) - \frac{1}{2}\rho_B(t,u)\right|^2 dk = 0$$

where

$$\partial_t \rho_B(t,u) = -\left(-\Delta\right)^{\frac{\alpha_B}{2}}\rho_B(t,u)$$

and

$$\frac{\alpha_B}{2} = \frac{5}{6} \text{ if } B\neq 0 \text{ and } \frac{\alpha_B}{2} = \frac{3}{4} \text{ if } B = 0$$

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$$d\nu_\delta(r) = \begin{cases} |r|^{-\frac{3}{2}-1}dr \text{ if } \delta > \frac{1}{2} & \longleftrightarrow -(-\Delta)^{\frac{3}{4}} \\ h_B(r)dr \text{ if } \delta = \frac{1}{2} & \longleftrightarrow \text{Generator of a (non-stable) Lévy process} \\ |r|^{-\frac{5}{3}-1}dr \text{ if } \delta < \frac{1}{2} & \longleftrightarrow -(-\Delta)^{\frac{5}{6}} \end{cases}$$

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Theorem (Cane): $N^{-1}Z_{Nu}^{B_N}\left(N^{\alpha_\delta}\,\cdot\,\right)$ converges to the Lévy process $Y_u^\delta(\,\cdot\,)$. with Lévy measure ν_δ where

$$\alpha_\delta = \frac{5-\delta}{3} \text{ if } \delta < \frac{1}{2} \text{ and } \alpha_\delta = \frac{3}{2} \text{ if } \delta \geq \frac{1}{2}$$

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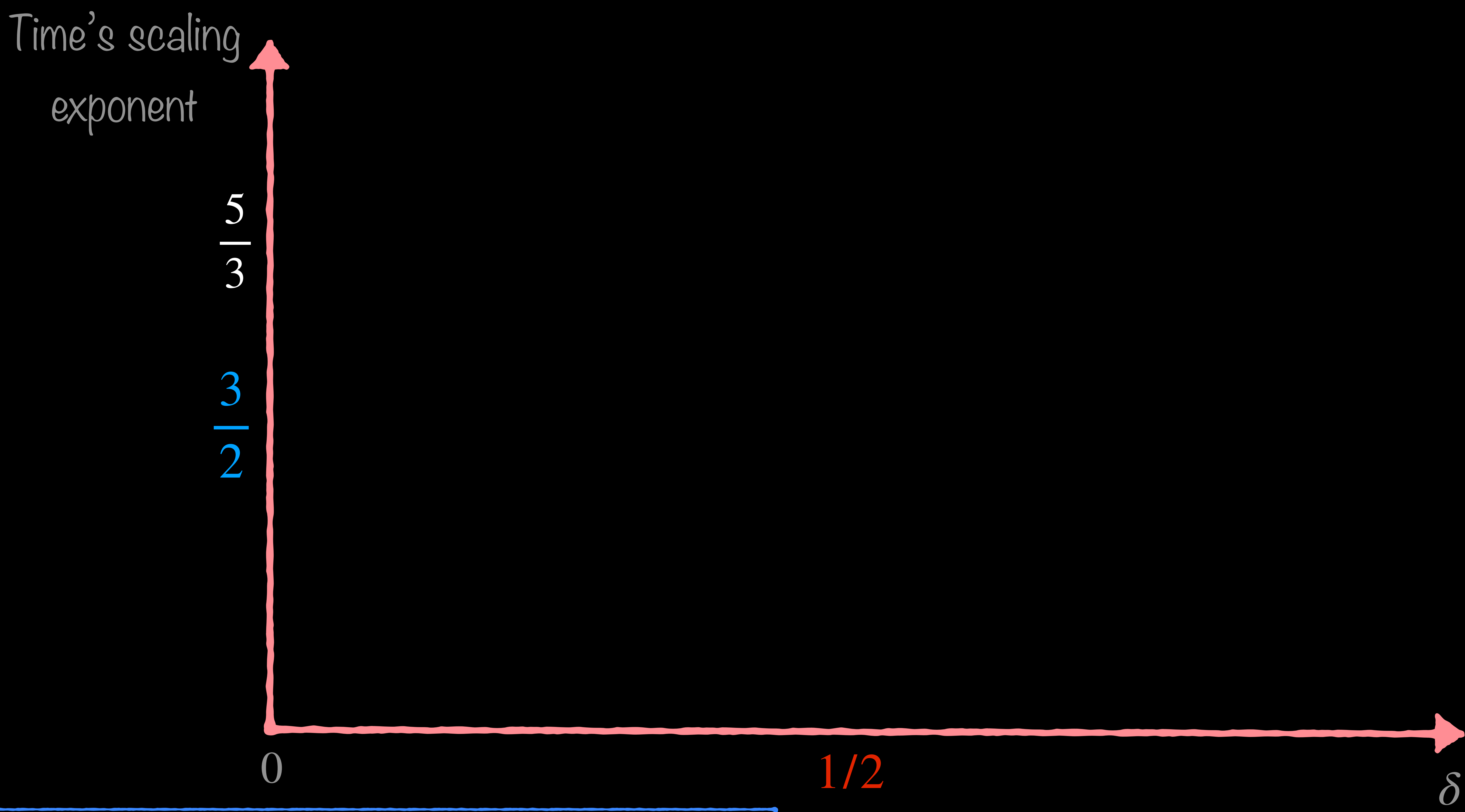

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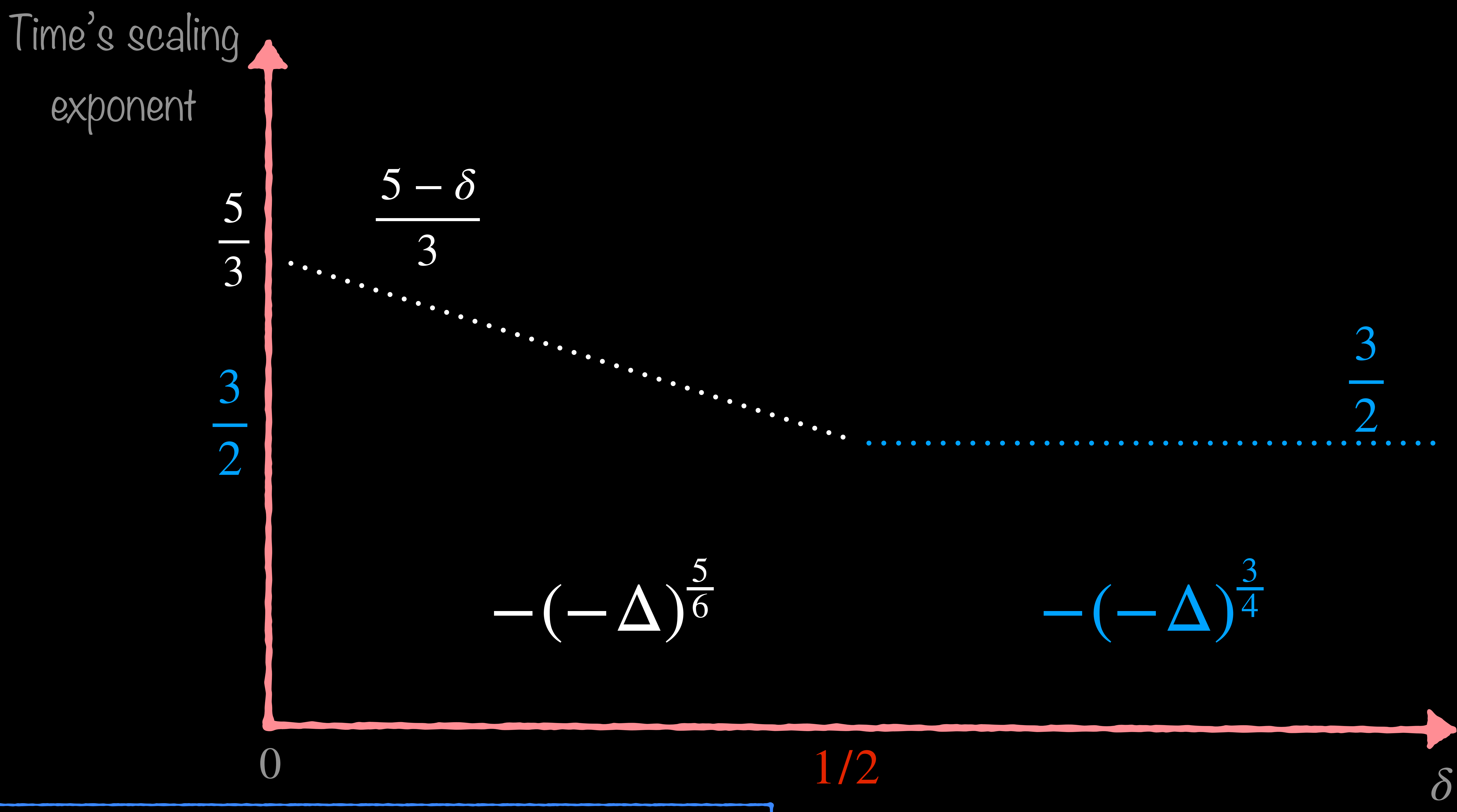

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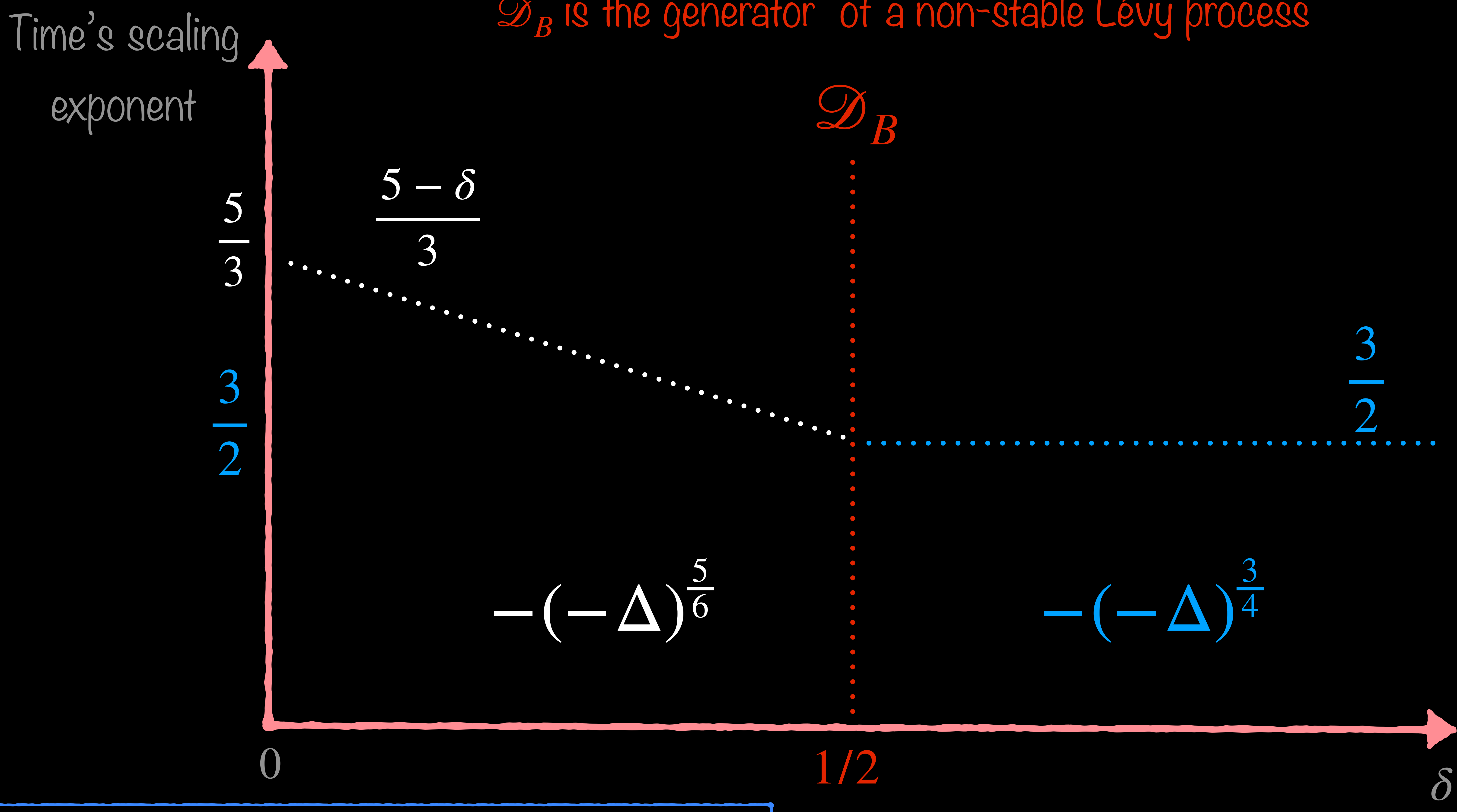

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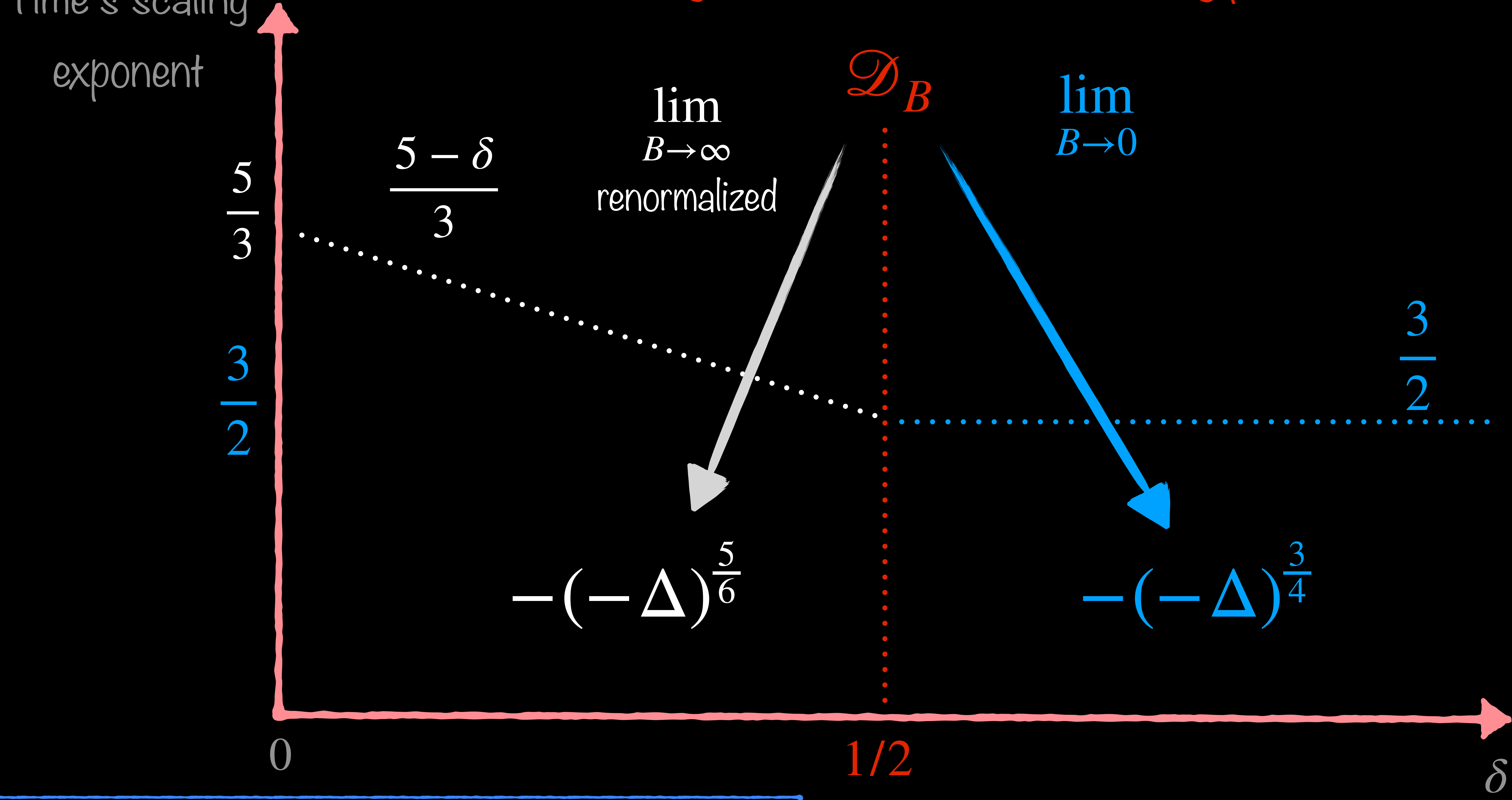

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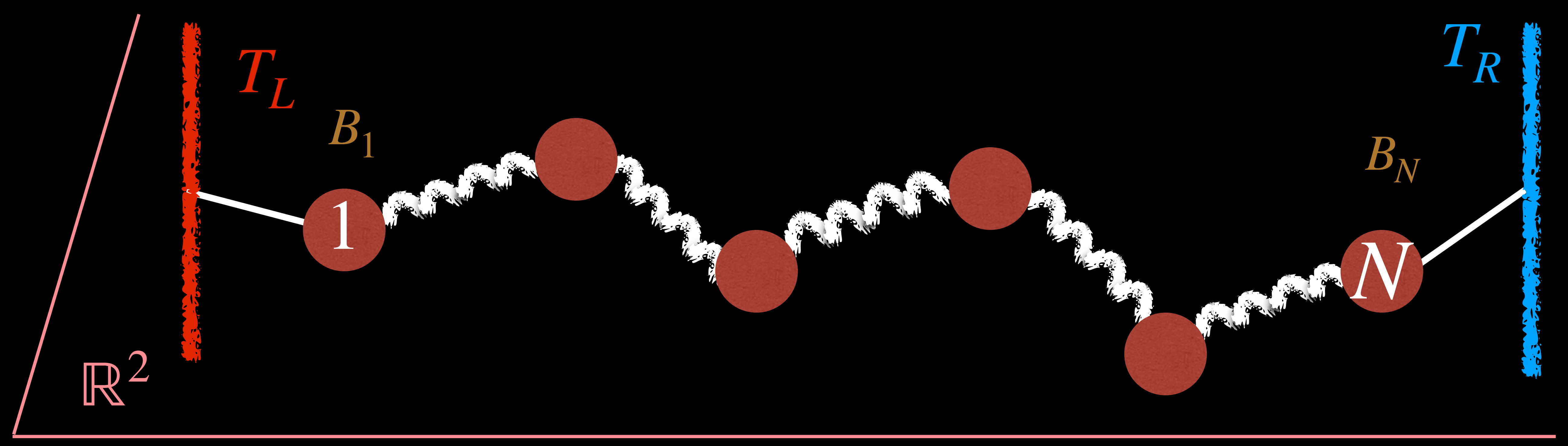
Part II
Study of the harmonic chain submitted to a random magnetic field

Work in collaboration with Cédric Bernardin, Junaid Majeed Bhat and Abhishek Dhar.

Harmonic chain submitted to a random magnetic field



$$dv_i(t,x) = \left[q_i(t,x+1) + q_i(t,x-1) - 2q_i(t,x)\right] dt$$
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In this presentation we work with fixed boundary conditions

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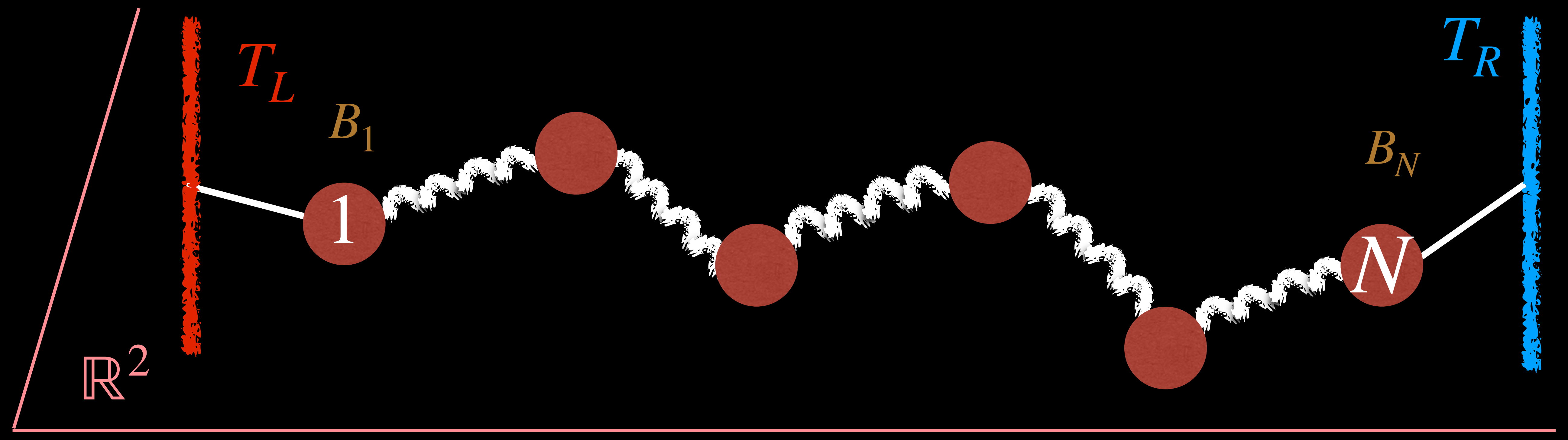

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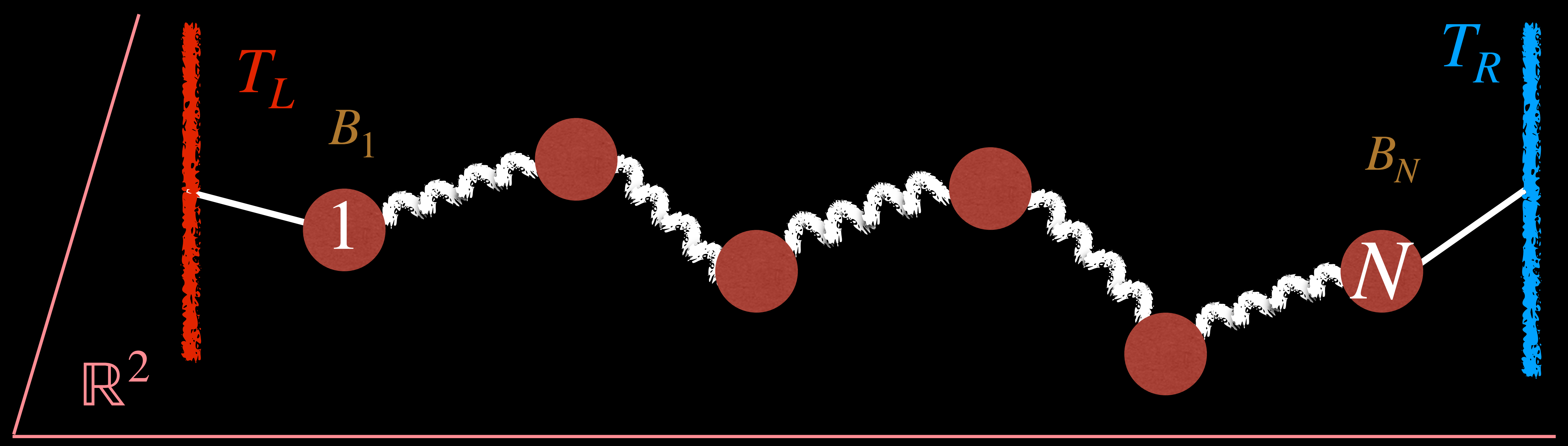

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$$f_{n+1}^\pm = \left(2 - \omega^2 \pm \omega B_{n+1}\right) f_n^\pm - f_{n-1}^\pm \quad g_{n+1}^\pm = \left(2 - \omega^2 \pm \omega B_{n+1}\right) g_n^\pm - g_{n-1}^\pm$$

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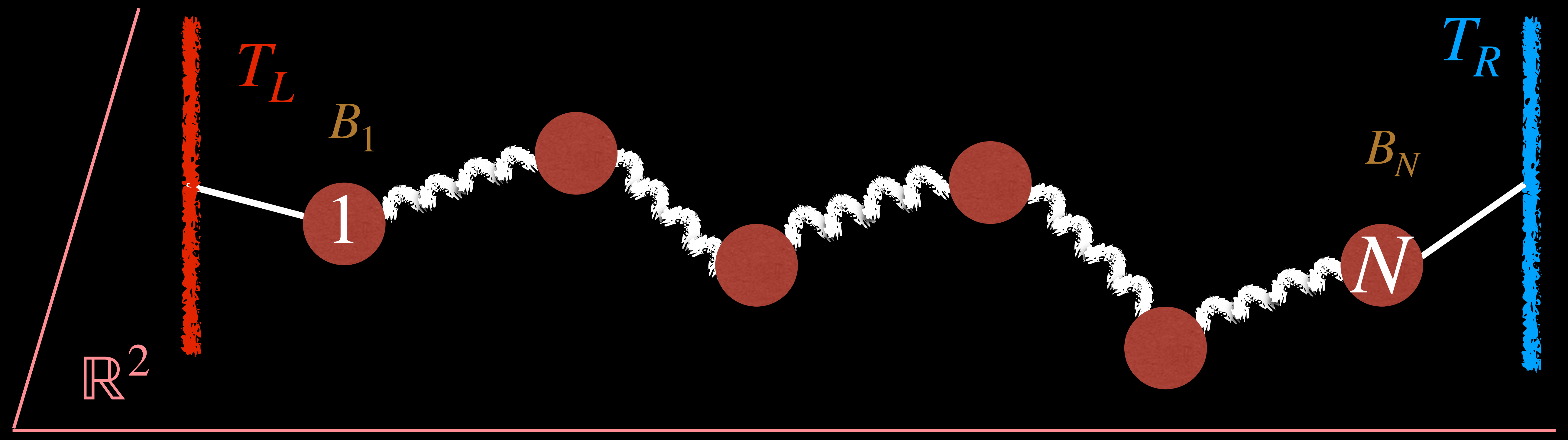

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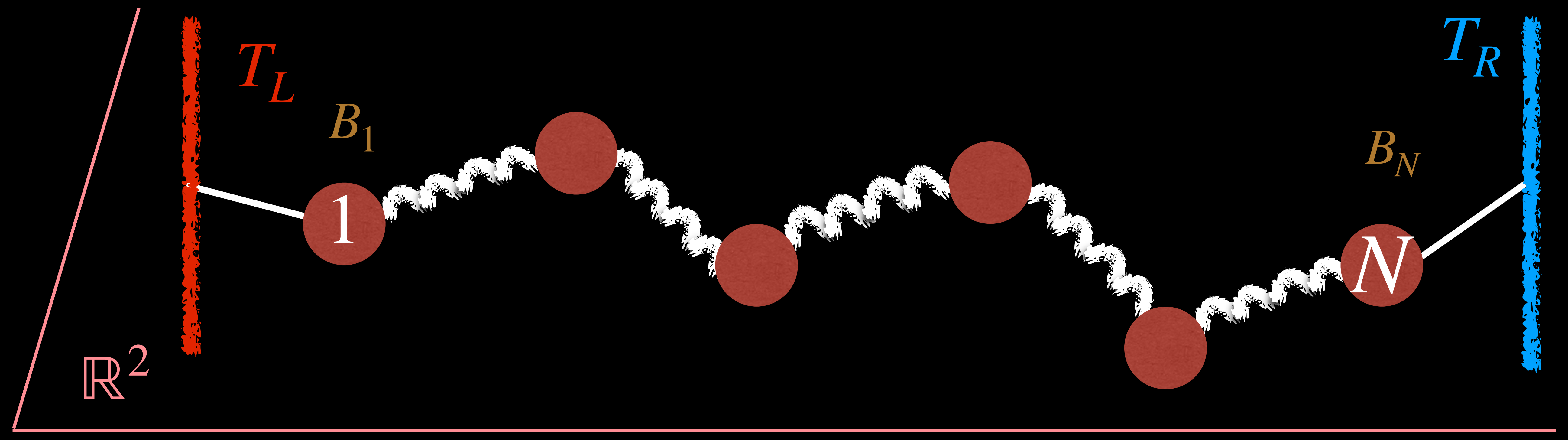
In this presentation we work with fixed boundary conditions

Bhat-**Cane**-Bernardin-Dhar [JSP'21] proved that

$$\langle J_N \rangle = \frac{4\left(T_L - T_R\right)}{\pi} \int_0^\infty \mathscr{T}_N(\omega) d\omega \quad \langle \cdot \rangle \text{ NESS} \quad \langle J_N \rangle \sim N^a \text{ with } a = ?$$

Harmonic chain submitted to a random magnetic field



$$dv_i(t,x) = \left[q_i(t,x+1) + q_i(t,x-1) - 2q_i(t,x)\right] dt$$
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For $B_n = B$

Bhat-**Cane**-Bernardin-Dhar [JSP'21]

If $B \neq 0$, then $\mathscr{T}_\infty(\omega) \sim \omega^{3/2}$ and If $B = 0$ then $\mathscr{T}_\infty(\omega) \sim \omega^2$ ⟶ $\langle J_N \rangle \sim T_L - T_R$

Formal explanation of the phenomenon

$$\mathscr{T}_N(\omega) = \frac{\omega^2}{\left| f_N^+ + \mathbf{i}\omega\left(g_N^+ + f_{N-1}^+\right) - \omega^2 g_{N-1}^+ \right|^2} + \frac{\omega^2}{\left| f_N^- + \mathbf{i}\omega\left(g_N^- + f_{N-1}^-\right) - \omega^2 g_{N-1}^- \right|^2}$$

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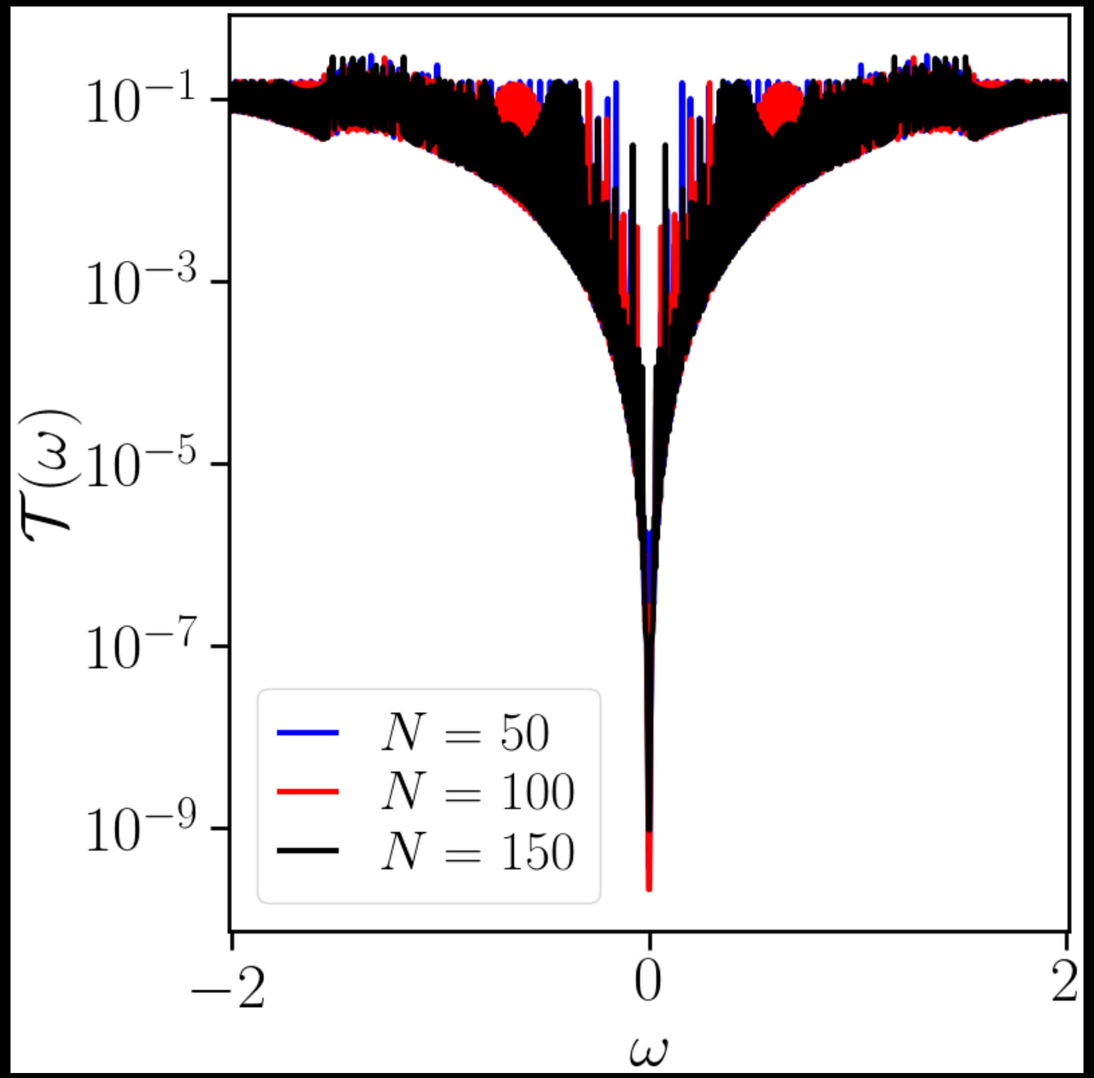
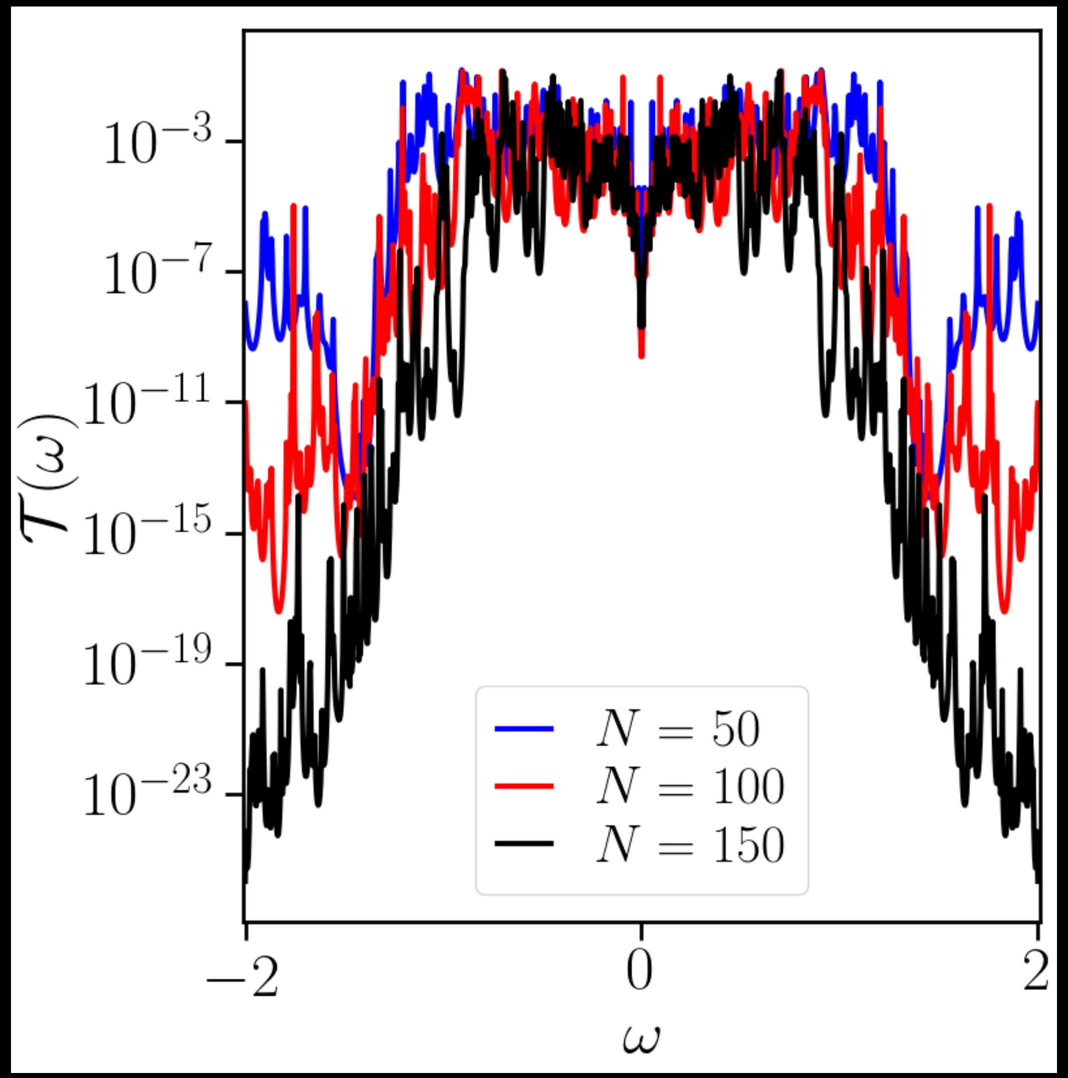
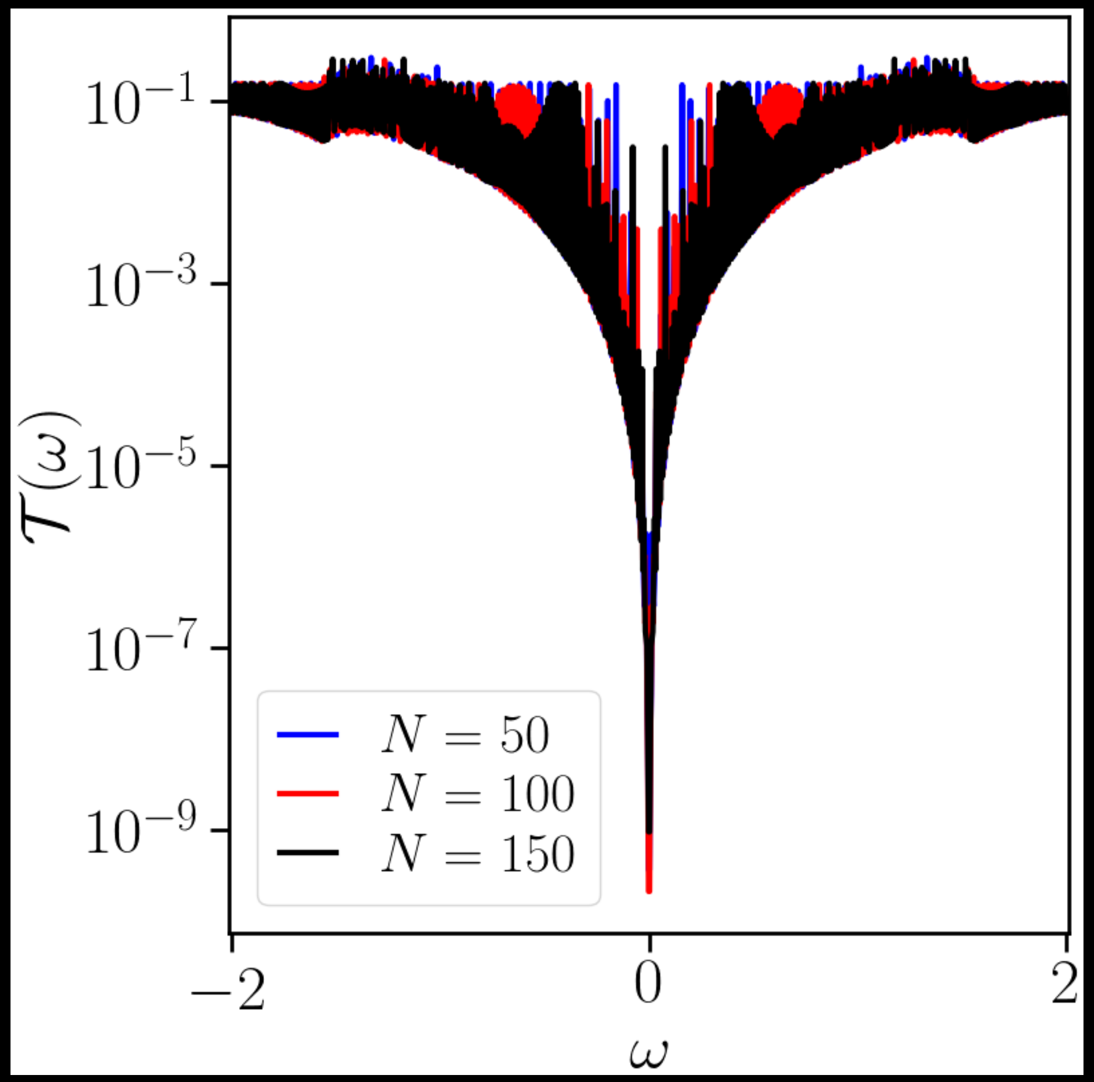


Constant magnetic field



Random magnetic field

Formal explanation of the phenomenon



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$$\mathcal{T}_N(\omega) \sim \omega^2 \exp\left[-2\lambda(\omega)N\right]$$

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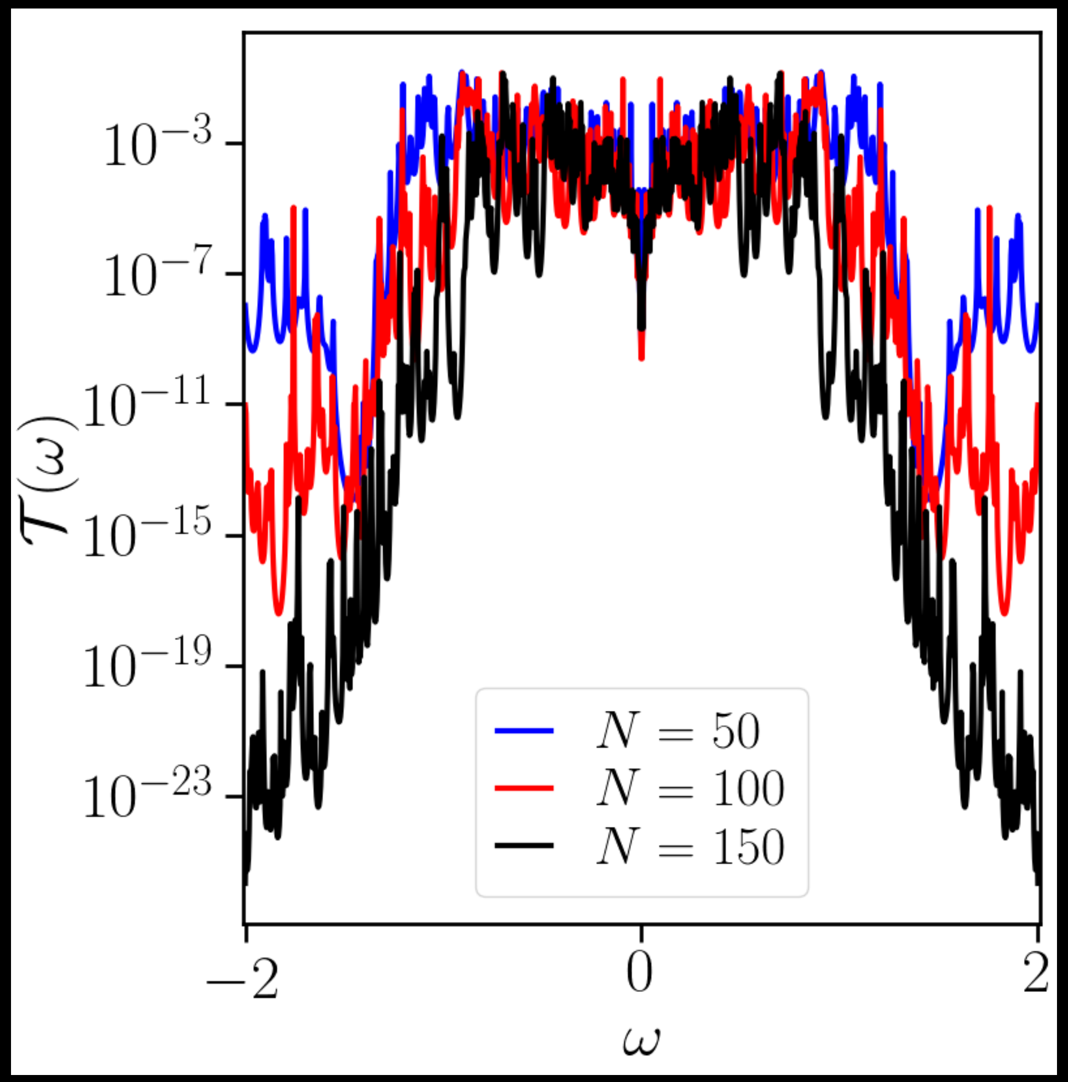

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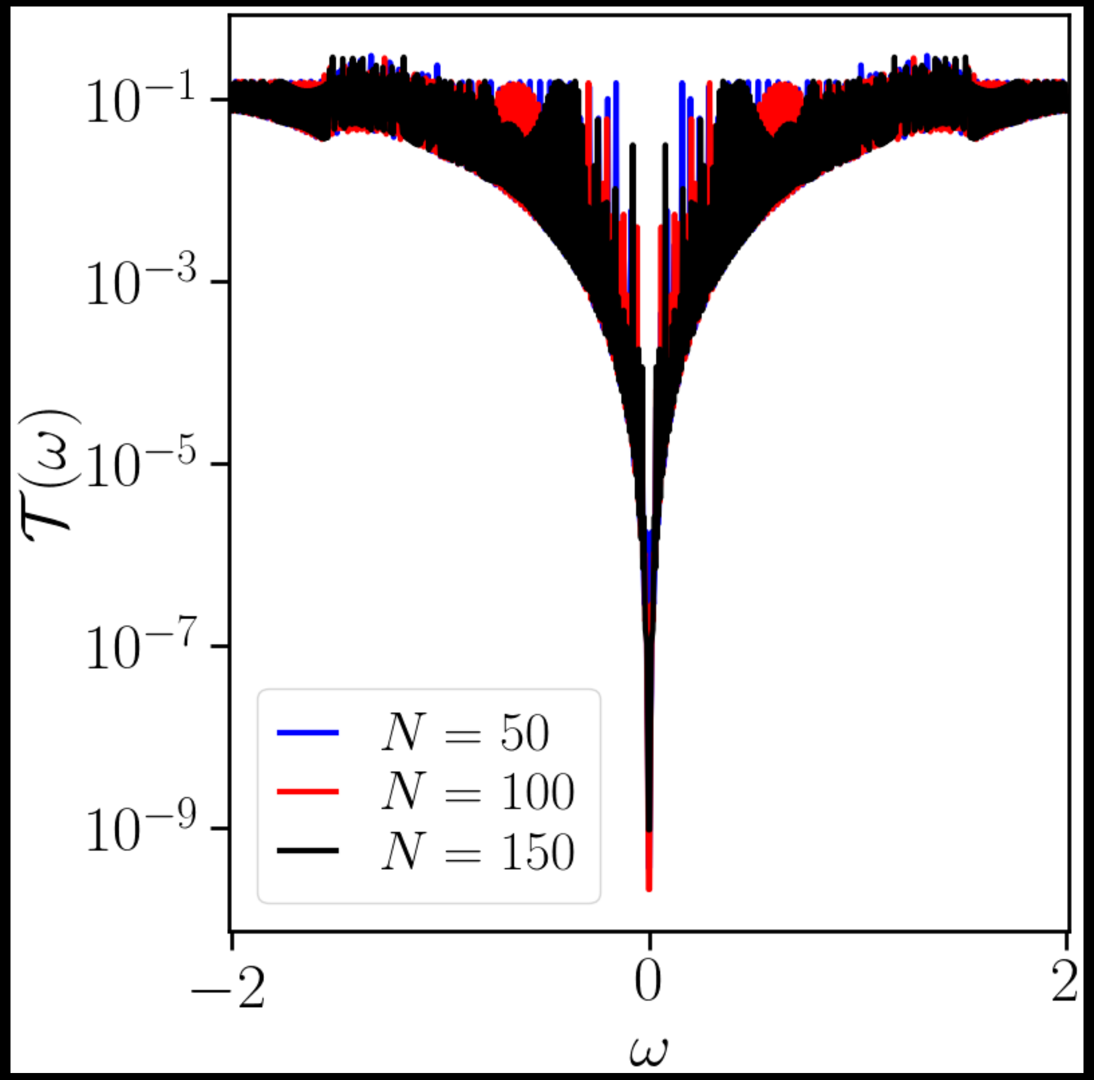

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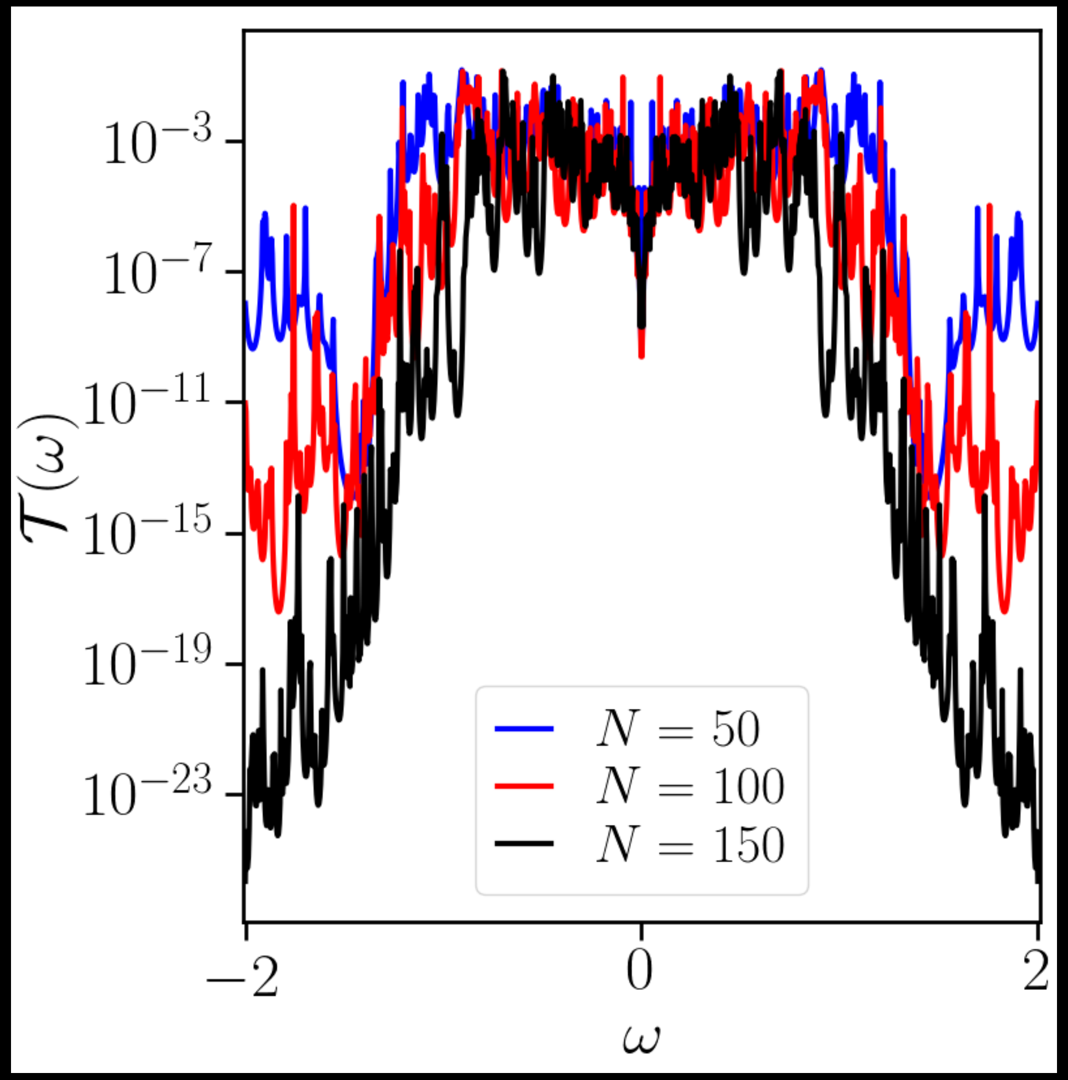

Random magnetic field

$$\mathbb{E}\left[\langle J_N \rangle\right] = \frac{4\left(T_L - T_R\right)}{\pi}\int_0^{\lambda^{-1}\left(\frac{1}{N}\right)} \mathbb{E}\left[\mathcal{T}_N(\omega)\right] d\omega \;+\; \frac{4\left(T_L - T_R\right)}{\pi}\int_{\lambda^{-1}\left(\frac{1}{N}\right)}^{\infty} \mathbb{E}\left[\mathcal{T}_N(\omega)\right] d\omega$$

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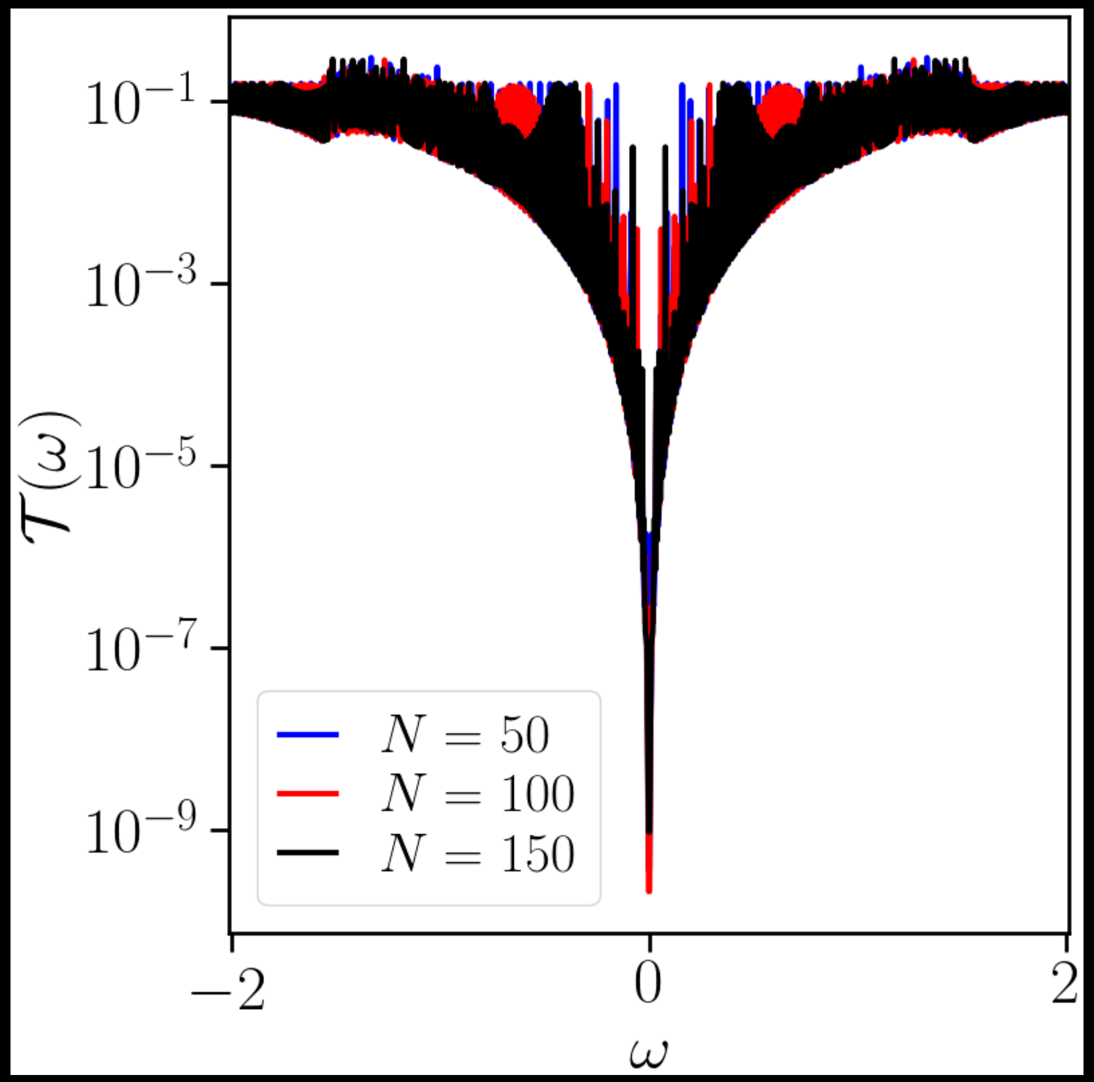

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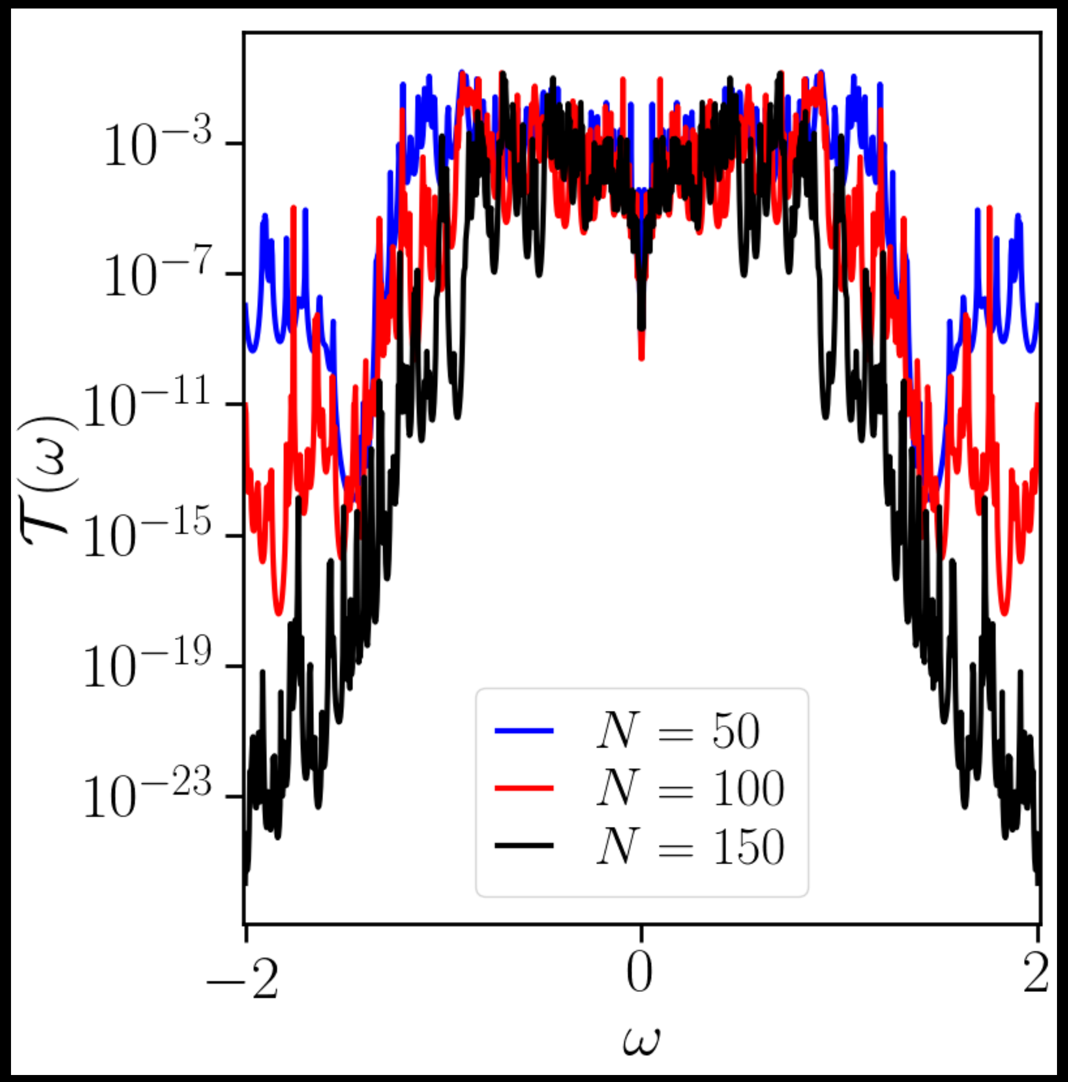

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$$\sim \frac{4\,(T_L - T_R)}{\pi}\int_0^{\lambda^{-1}\left(\frac{1}{N}\right)} \mathcal{T}_\infty(\omega) d\omega$$

$\mathcal{T}_\infty$ is the transmission for a harmonic chain submitted to a constant magnetic field $\mathbb{E}[B]$.

Lyapunov exponent of $f_n^{\pm}$

Let $\sigma^2 = \mathbb{V}\left(B_0\right)$ and ξ be a white noise.

$$f_{n+1}^{\pm} = \left(2 - \omega^2 \pm \omega B_{n+1}\right) f_n^{\pm} - f_{n-1}^{\pm} \quad \xleftrightarrow{\text{Some kind of miracle}} \quad \ddot{f}^{\pm}(t) = \pm\, \omega \mathbb{E}\left[B\right] f^{\pm}(t) - \omega\sigma\xi(t) f^{\pm}(t)$$

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Cane-Bhat-Dhar-Bernardin [JSM'22]

$\tilde{\lambda}^{-}(\omega) \sim \sqrt{\left|\mathbb{E}\left[B\right]\right|}\,\omega^{1/2}$ if $\mathbb{E}\left[B\right] < 0$ and $\tilde{\lambda}^{-}(\omega) \sim C\omega^{2/3}$ if $\mathbb{E}\left[B\right] = 0$ and $\tilde{\lambda}^{-}(\omega) \sim \frac{\sigma^2}{8\mathbb{E}\left[B\right]}\omega$ if $\mathbb{E}\left[B\right] > 0$

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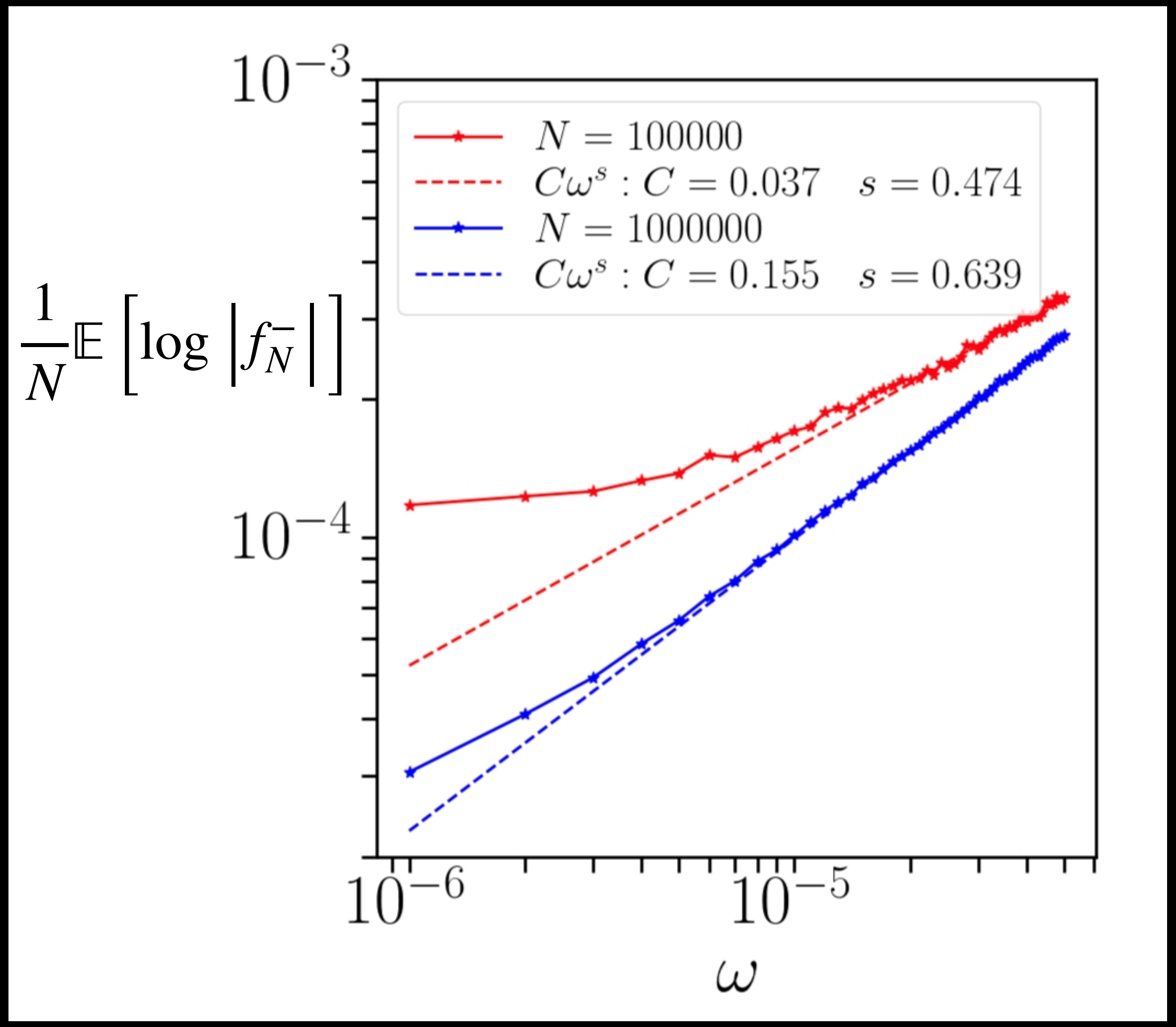
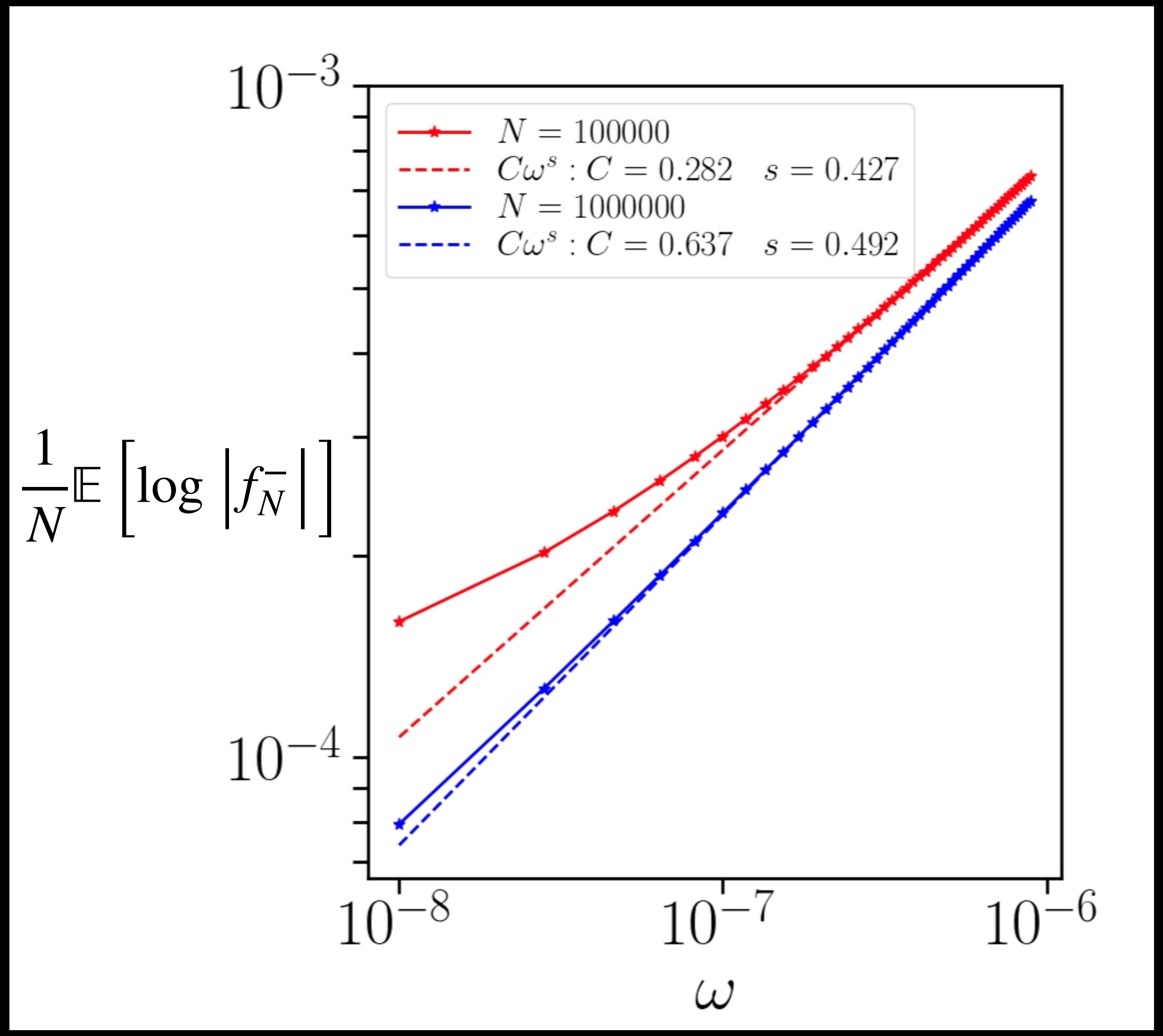
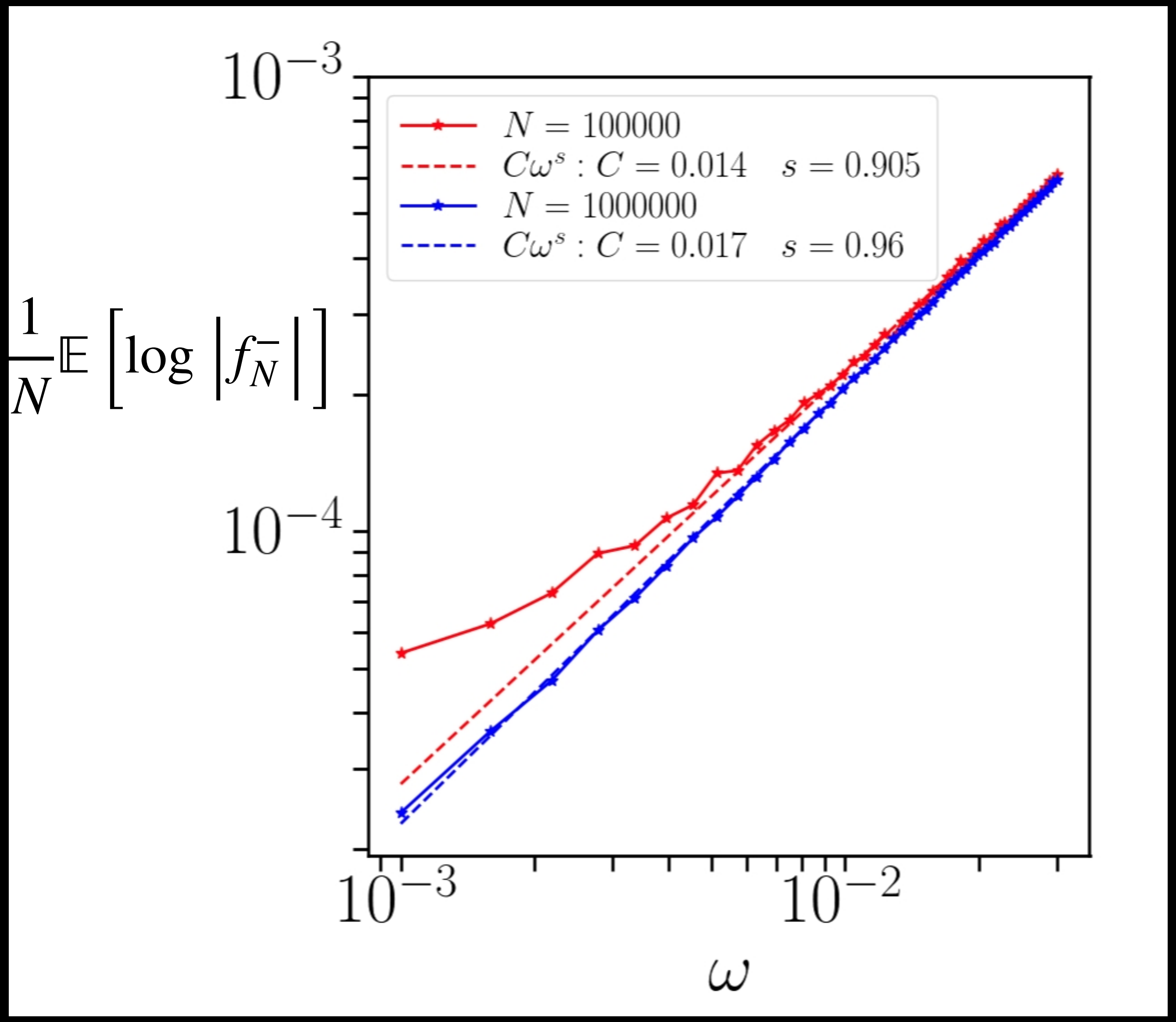


$\mathbb{E}[B] < 0$



$\mathbb{E}[B] = 0$



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Cane-Bhat-Dhar-Bernardin [JSM'22]

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Size of the heat current

$$\mathbb{E}\left[\left\langle J_N \right\rangle\right] \sim \frac{4\left(T_L - T_R\right)}{\pi} \int_0^{\lambda^{-1}\left(\frac{1}{N}\right)} \mathcal{T}_\infty(\omega) d\omega$$

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- If $\mathbb{E}[B] = 0$, then $\mathbb{E}\left[\langle J_N \rangle\right] \sim N^{-9/2}$ $\rightarrow \kappa \sim N^{-7/2}$

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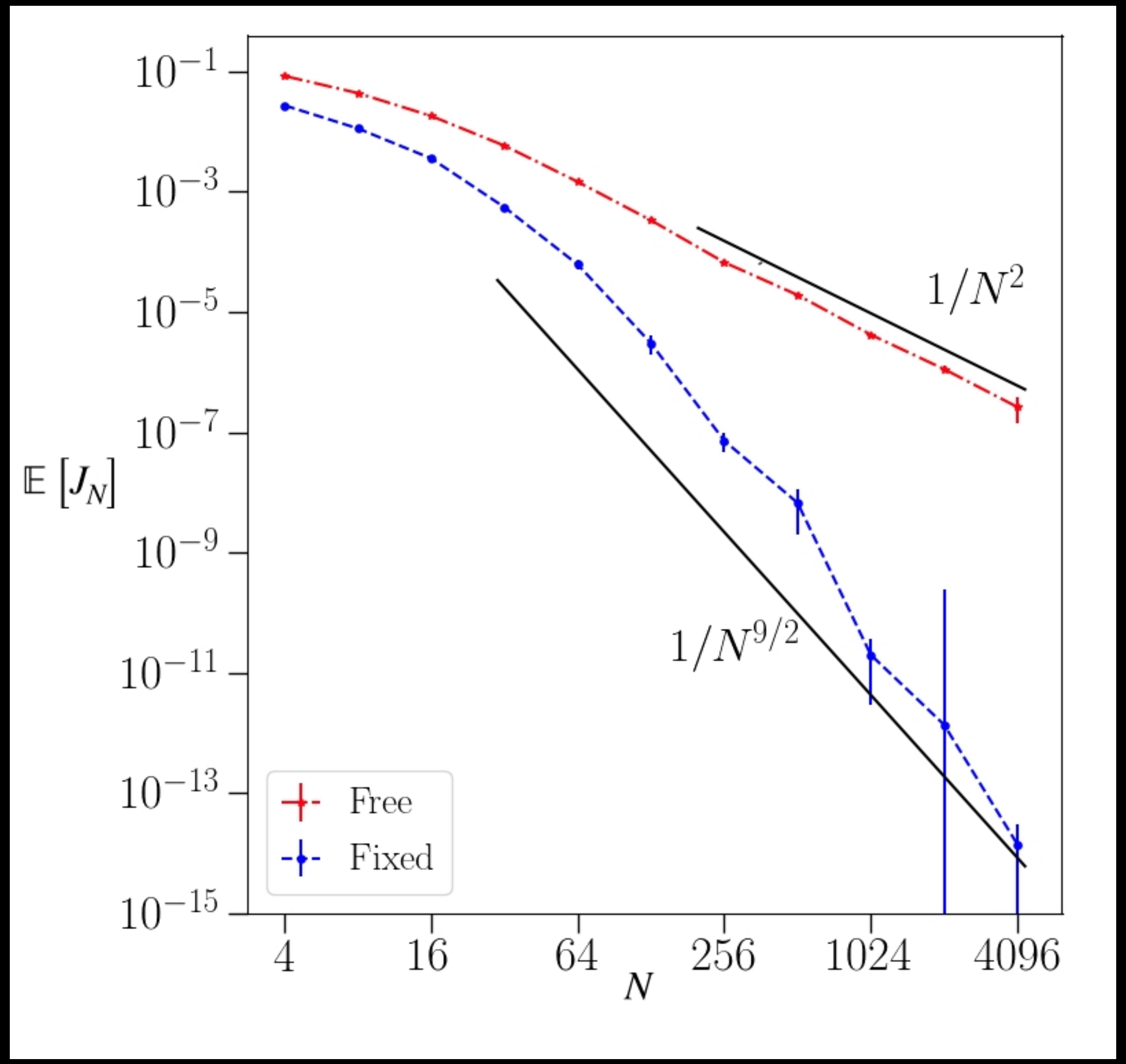

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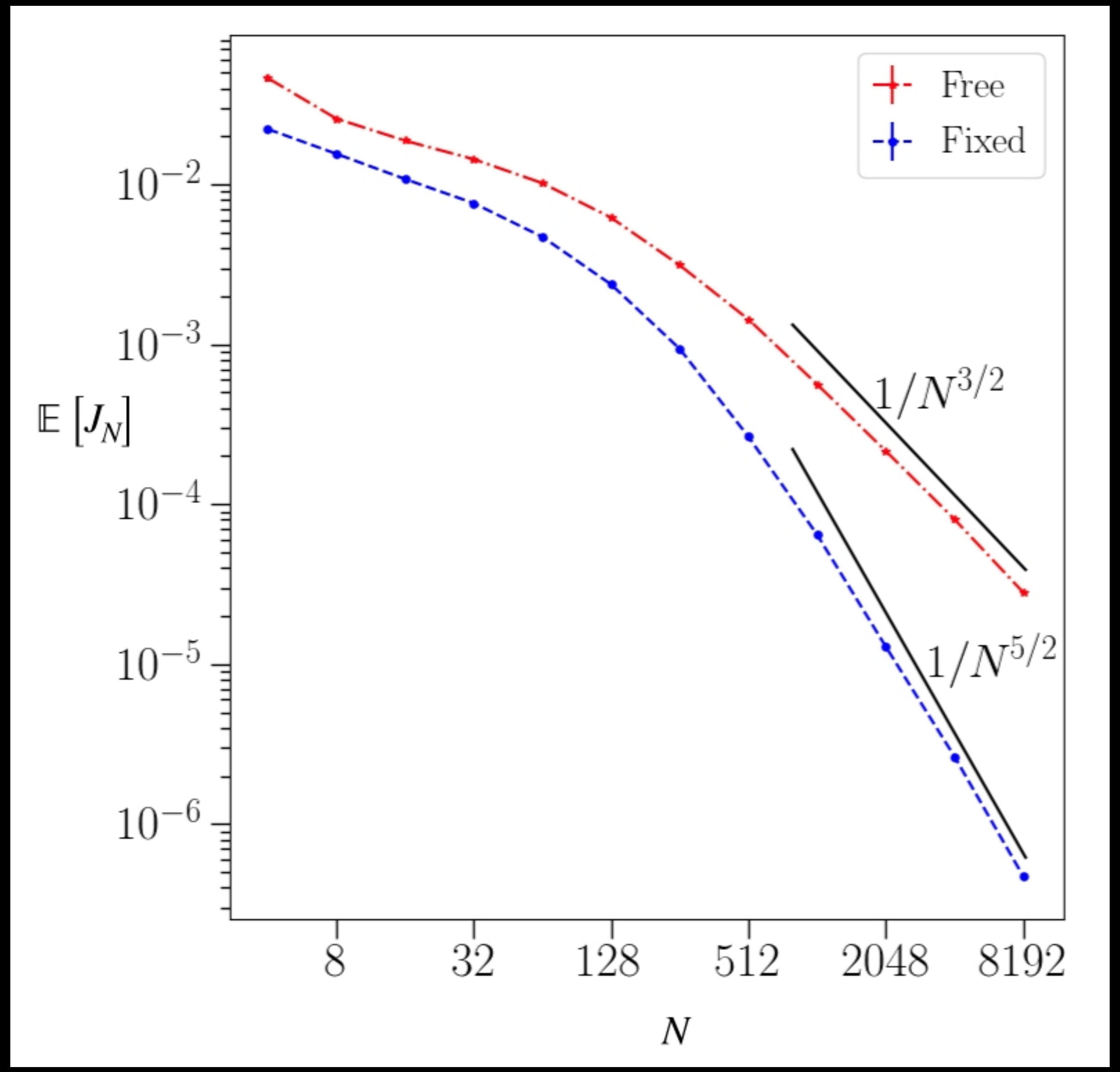

$\mathbb{E}[B] \neq 0$

Numerical simulations don't match when $\mathbb{E}[B] = 0$

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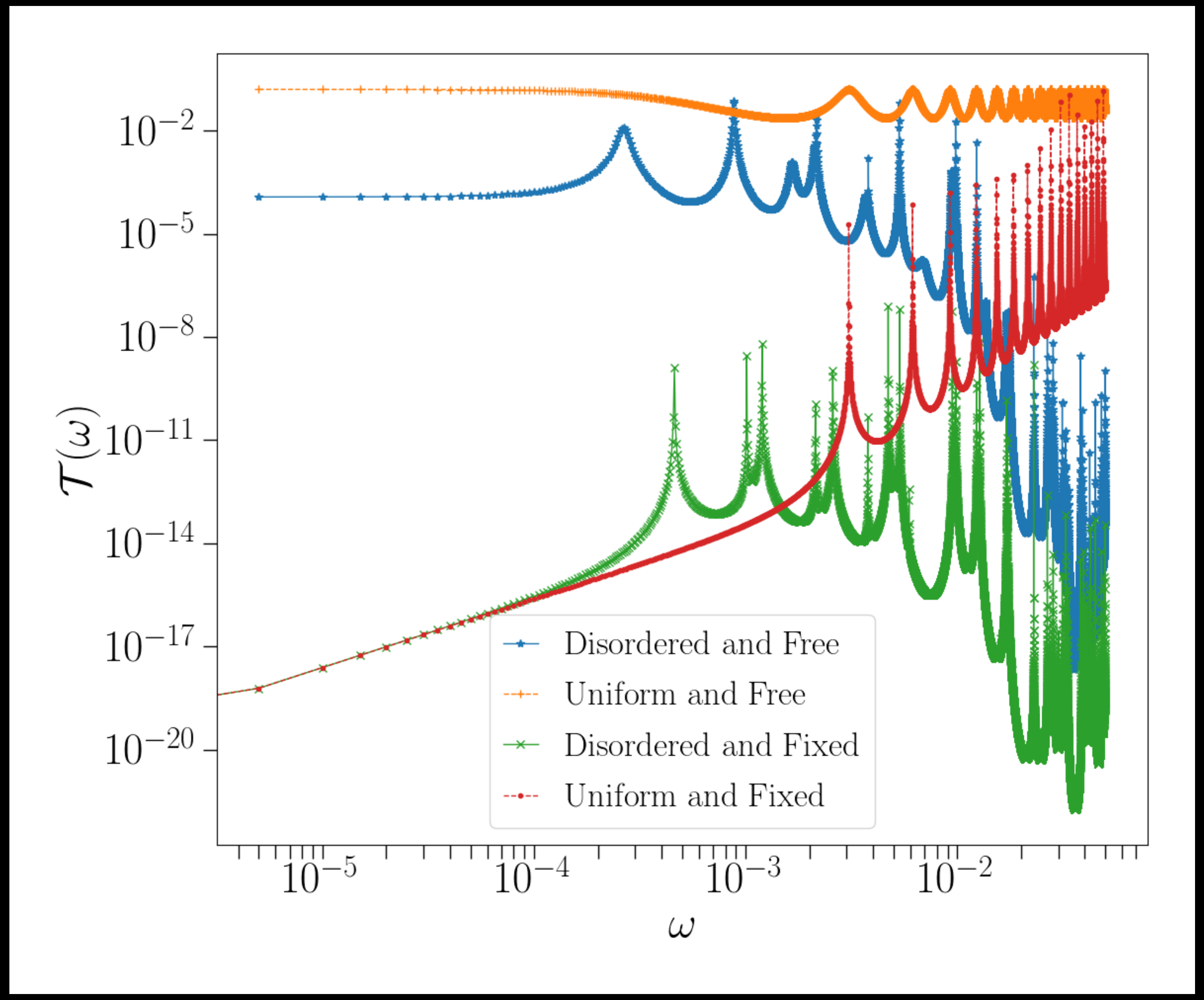
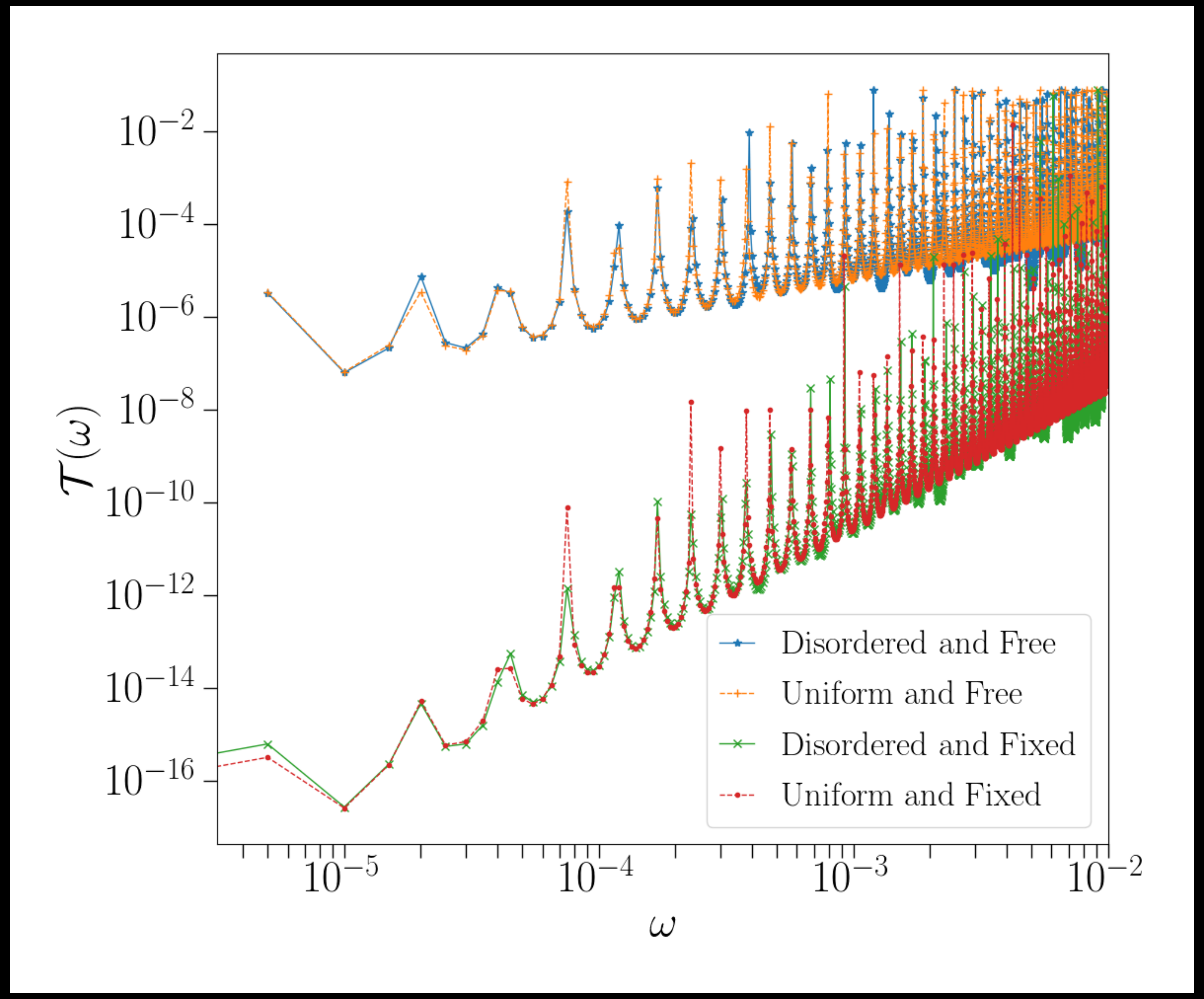


$\mathbb{E}[B] = 0$

The approximation $\mathbb{E}\left[\mathcal{T}_N\right] \sim \mathcal{T}_\infty$ seems to be false for $\mathbb{E}[B] = 0$



$\mathbb{E}[B] \neq 0$

Lowest allowed normal modes $\sim \lambda^{-1}(\omega)$

Summary and open problems

Brief summary of the results obtained

- Transition between two fractional diffusion equations depending on the intensity of the magnetic field [C **submitted**]
- Study of the heat current for an ordered chain submitted to a constant magnetic field [BCBD JSP'22]
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Some open problems

- Transition in one step (work in progress)
- Rigorous study of the heat current for a random magnetic field
- Transition between two fractional Laplacian with a magnetic interface (work in progress with **Simon**)

THANK YOU FOR
YOUR ATTENTION