Superdiffusion transition for a noisy harmonic chain subject to a magnetic field

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Nice
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Introduction to the different scales

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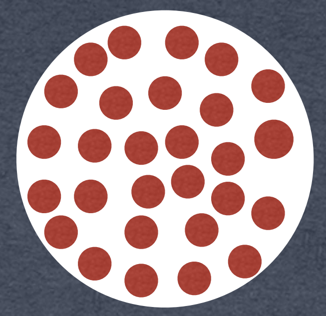
Microscopic scale

System composed of many particles and described by Newton's Laws



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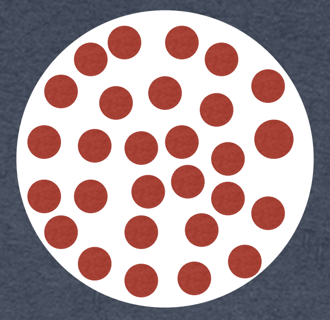

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System described by a kinetic equation on the density f

$$\partial_t f + v \nabla_x f = Q[f, f]$$

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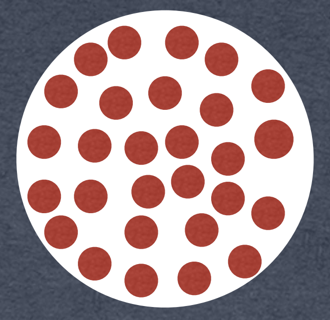

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- Euler's equations
- Diffusion equation

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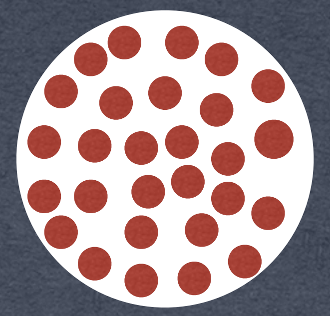

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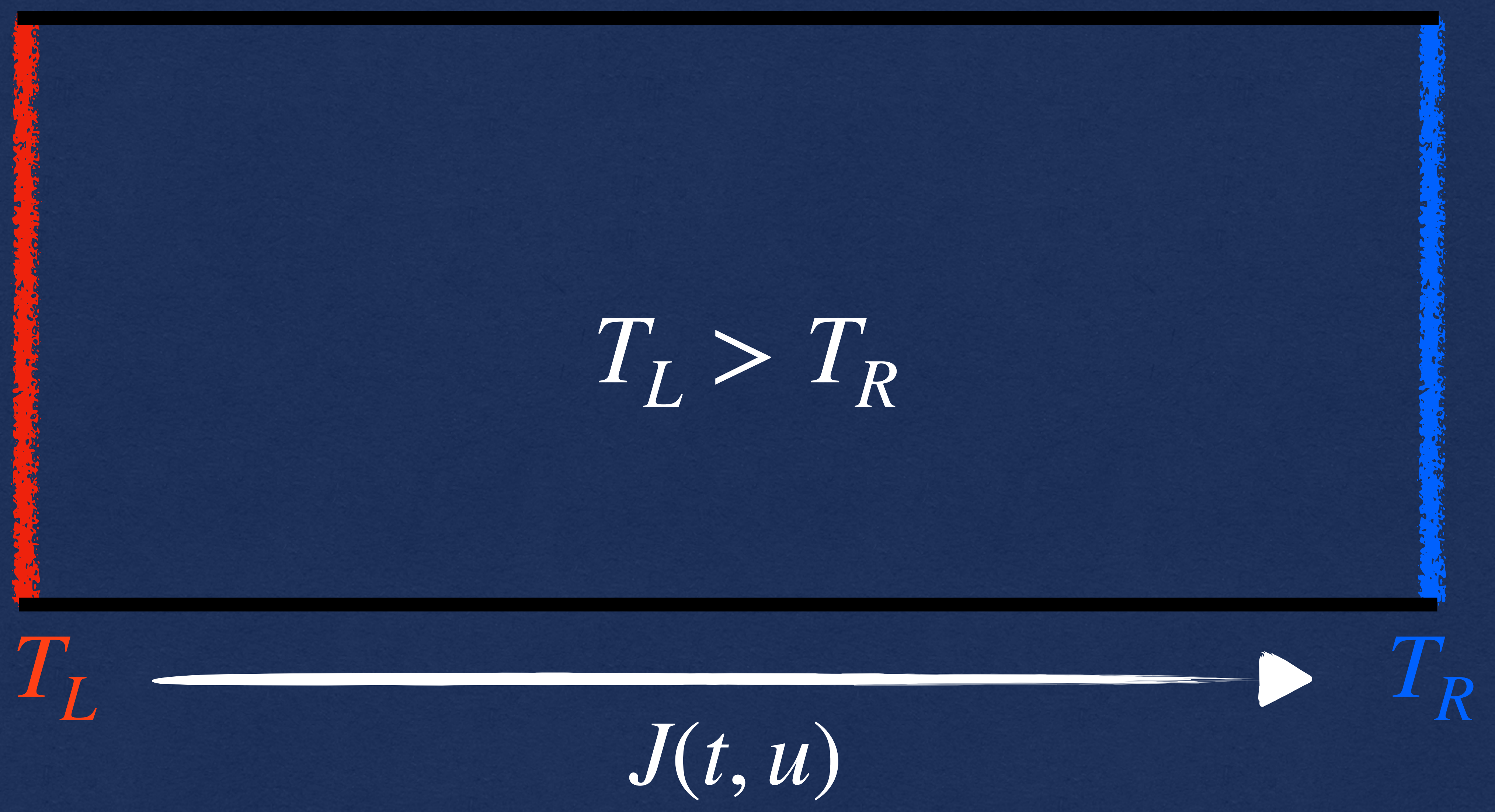
Microscopic → Macroscopic: **Hydrodynamic limits** One-step

Mesoscopic → Macroscopic: **Hydrodynamic limits** Two-steps

Fourier's law and diffusion equation

1822 : Fourier's experimental law



$$J(t, u) = -\kappa(T)\,\nabla T(t, u)\,.$$

Fourier's law and diffusion equation

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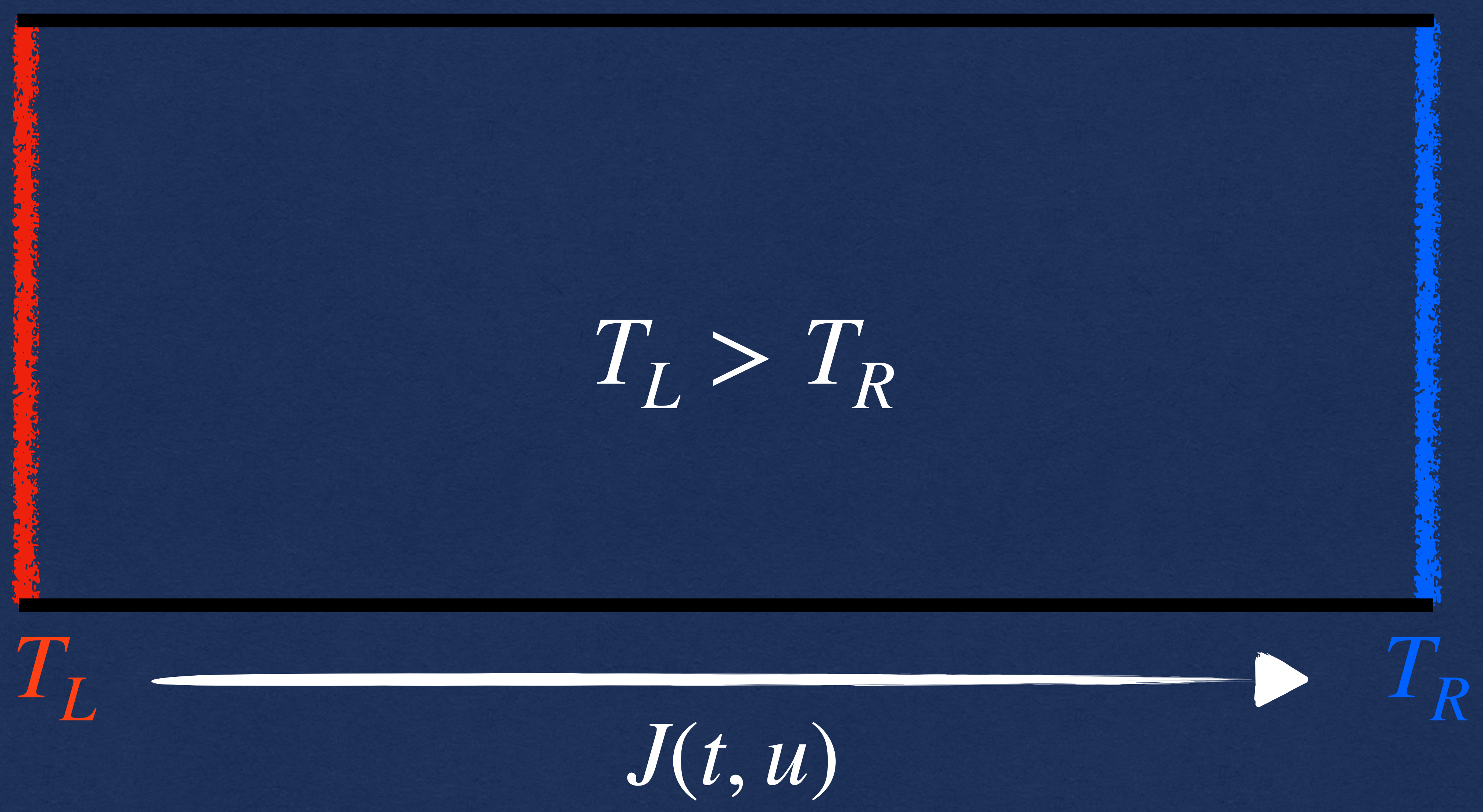

$$J(t,u) = -\kappa(T)\,\nabla T(t,u)\,.$$

This leads to

$$\partial_t T(t,u) = \nabla_u\big(\kappa(T)\,\nabla_u T(t,u)\big)\,.$$

Diffusion equation

FPU chains

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$$x \in \{1, \ldots, N\} \qquad \frac{d}{dt} q(t,x) = v(t,x)$$

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Energy of the system

$$E(t) = \frac{1}{2} \sum_{x=1}^{N} m_x |v(t,x)|^2 + \frac{1}{2} \sum_{x=1}^{N} \left(q(t,x) - q(t,x-1)\right)^2 + \alpha W\left(q(t,x) - q(t,x-1)\right).$$

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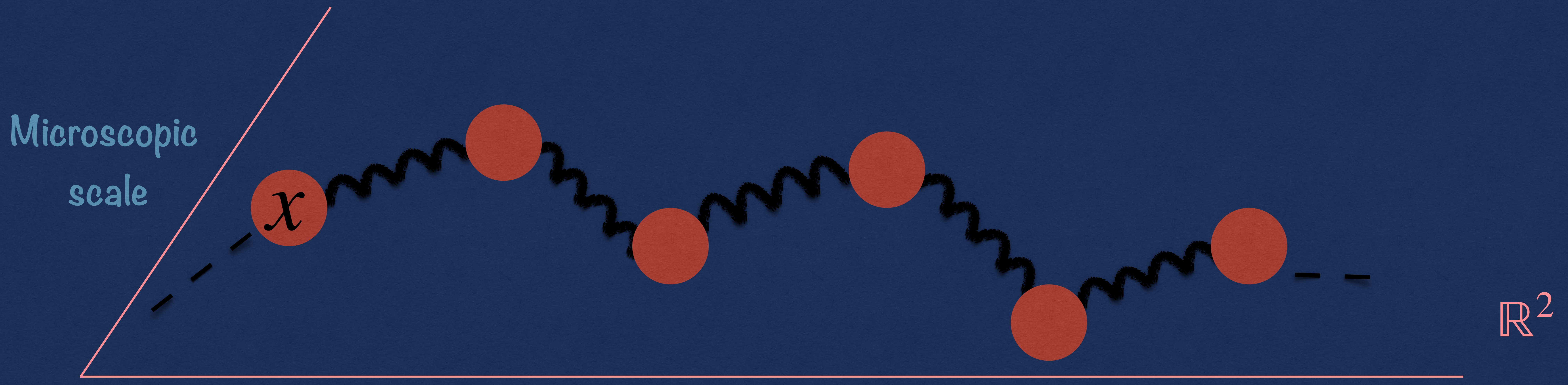
In one dimensional systems heat conductivity seems to be anomalous for $\alpha \neq 0$.

Harmonic chain submitted to a momentum conserving noise



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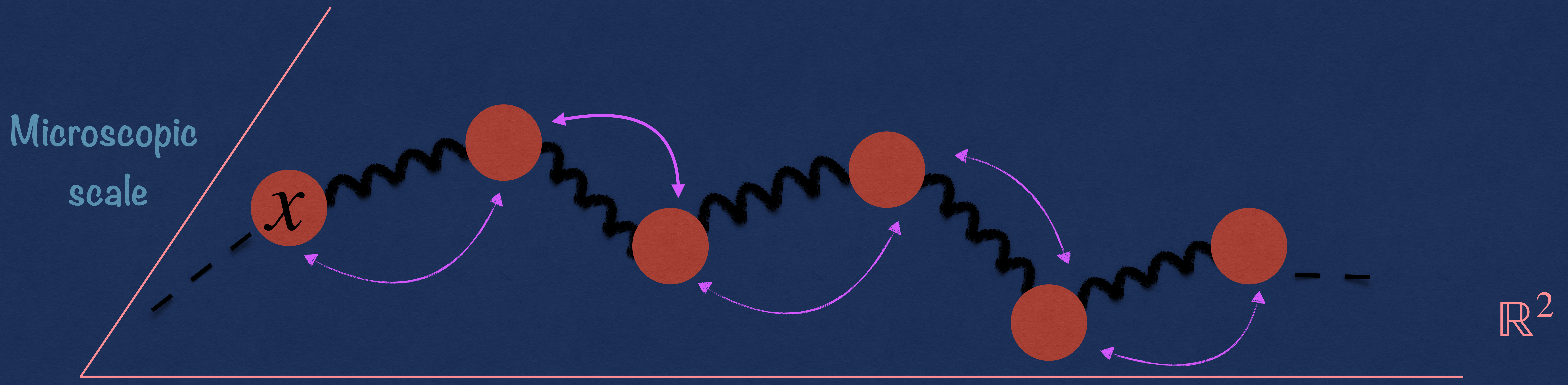

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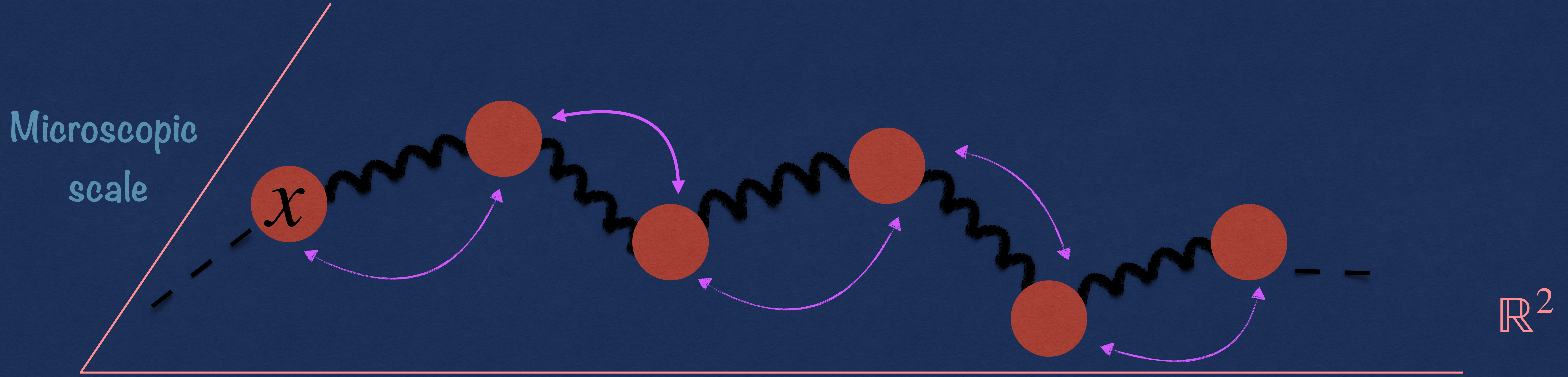

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The noise preserves the energy and the momentum .

$$E(t) = \sum_{x\in\mathbb{Z}} e(t,x)\,. \qquad\qquad P(t) = \left(\sum_{x\in\mathbb{Z}} v_1(t,x), \sum_{x\in\mathbb{Z}} v_2(t,x)\right).$$

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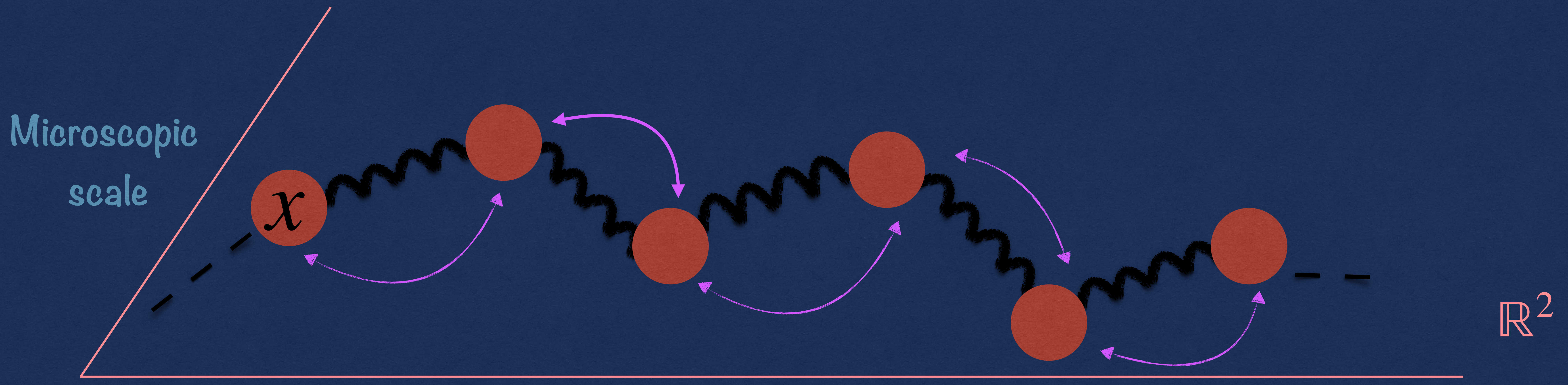

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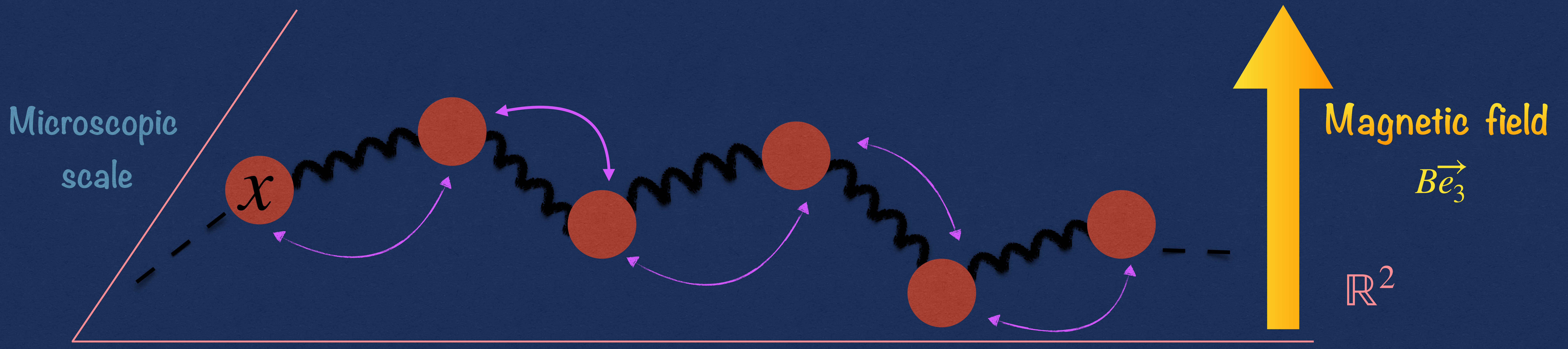

$$x \in \mathbb{Z} \quad \frac{d}{dt}v_i(t,x) = q_i(t,x+1) + q_i(t,x-1) - 2q_i(t,x) \quad +\varepsilon \ \textbf{noise}(t,x) \quad +B(\delta_{i,1}v_2(t,x) - \delta_{i,2}v_1(t,x))$$

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- SSS [CMP' 19] add a magnetic field of intensity B to the deterministic system.

Aim of the study

Let μ^ε be the initial distribution of the system. $\varepsilon \to 0$

Natural assumption :

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}[e(0,x)] = \int_{\mathbb{R}} J(u)\mathscr{W}_0(u)du\,.$$

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Let $J = (J_1, J_1)$ be a pair of functions independent of k then

$$\langle \mathscr{W}^\varepsilon(t), J\rangle = \varepsilon \sum_{x\in\mathbb{Z}} \mathbb{E}_{\mu^\varepsilon}\left[e\left(t\varepsilon^{-1}, x\right)\right] J_1(\varepsilon x) + \mathscr{O}_J(\varepsilon)\,.$$

To understand the behavior of the energy, we have to understand the one of $\mathscr{W}^\varepsilon$.

Historical results of BJKO

BOS [ARMA'10] and SSS [CMP'19] proved that $\mathscr{W}^{\varepsilon}$ converges to f where

$$\partial_t f(t,u,k,i) - \frac{\nu_B(k)}{2\pi}\partial_u f(t,u,k,i) = \mathscr{L}_B[f](t,u,k,i)\,.$$

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Mesoscopic scale

Here

$$\mathscr{L}_B f(t,u,k,i) = \sum_{j=1}^{2}\int_{\mathbb{T}} \theta_{i,B}^2(k)R(k,k')\theta_{j,B}^2(k')\,\big(f(t,u,k',j) - f(t,u,k,i)\big)dk'\,.$$

$$\mathbf{v}_B(k) = \frac{\sin(\pi k)\cos(\pi k)}{\sqrt{\sin^2(\pi k) + \frac{B^2}{4}}}$$

and

$$\theta_{1/2,B}^2 = \frac{1}{2} \pm \frac{B}{4\sqrt{\sin^2(\pi k) + \frac{B^2}{4}}}\,.$$

Introduction to Random Walks

$X \sim \mathcal{N}(m, \sigma^2)$ iff $\mathbb{P}(X \in A) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \displaystyle\int_A \exp\left(-\frac{|x-m|^2}{2\sigma^2}\right) dx\,.$

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Let $(X_n)_{n \in \mathbb{N}}$ a sequence of **i.i.d** random variables such that $\mathbb{P}\left(X_0 = 1\right) = \mathbb{P}\left(X_0 = -1\right) = \frac{1}{2}\,.$

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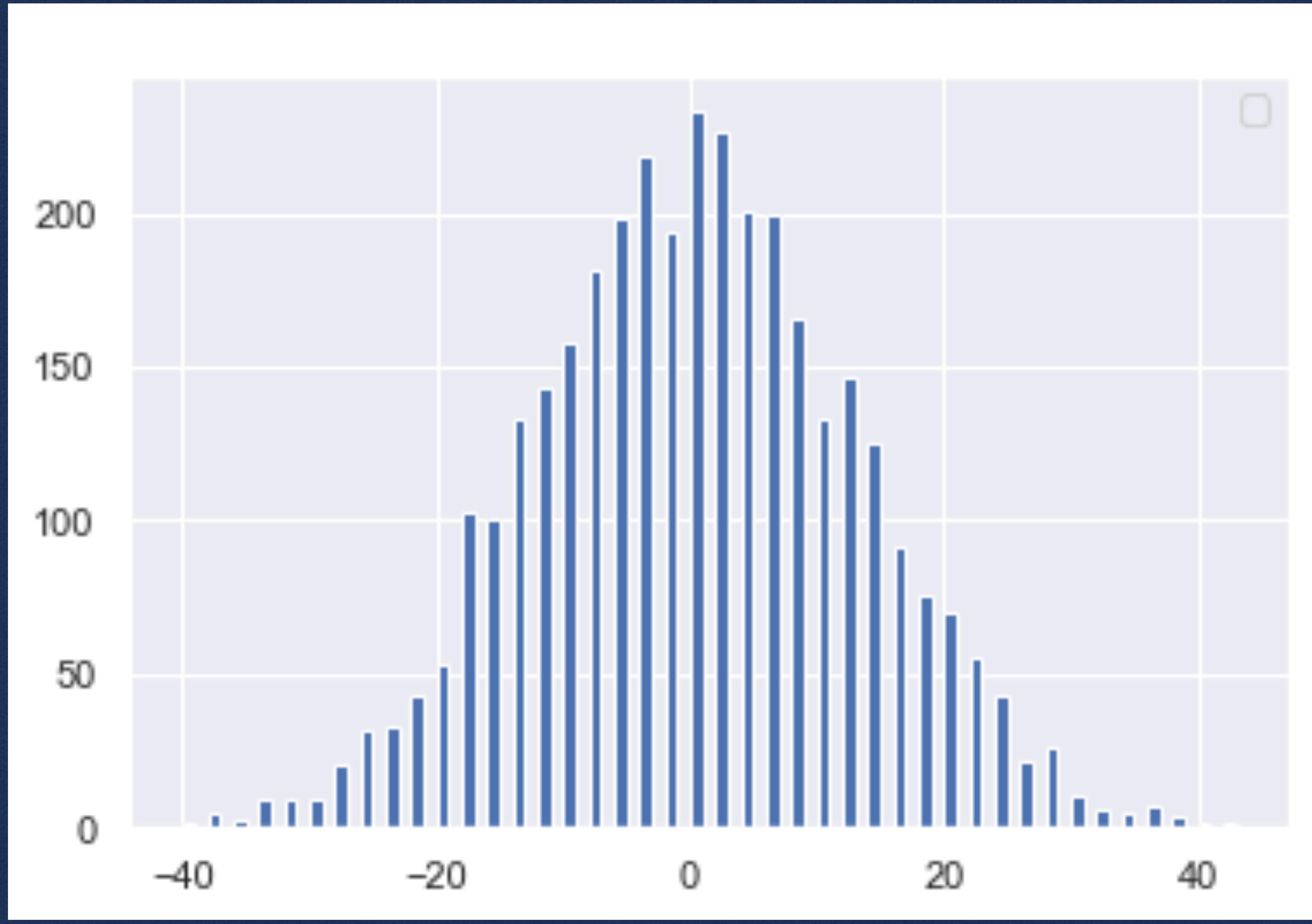

Brownian motion and Heat equation

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ρ is solution of

$$\partial_t \rho(t,u) = \frac{1}{2}\Delta[\rho](t,u)\,.$$

Brownian motion induces diffusion.

Lévy process

Lévy process

Let $\alpha \in (1,2)$ and σ a measure on $\mathbb{R}^*$ such that $d\sigma(r) = |r|^{-\alpha-1} dr$.

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Then

$$\partial_t \rho(t,u) = -(-\Delta)^{\frac{\alpha}{2}}[\rho](t,u) \longrightarrow \partial_t \rho(t,u) = \Delta[\rho](t,u)\,.$$

Levy process induces fractionnal diffusion.

A jump process

$$\partial_t f(t,u,k,i) - \frac{\nu_B(k)}{2\pi}\partial_u f(t,u,k,i) = \mathscr{L}_B[f](t,u,k,i)\,.$$

Mesoscopic scale

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We define a jump process $\big(K(\cdot), I(\cdot)\big)$

- $\big(K(0), I(0)\big) = (k,i)\,.$
- The process waits a time $\lambda_B(k,i)\,.$
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Then

$$f_i(t,u,k) = \mathbb{E}_{(k,i)}\left[f^0\left(Z_u(t), K(t), I(t)\right)\right] \longrightarrow f(t,u) = \mathbb{E}\left[f_0\left(\mathscr{B}_u(t)\right)\right].$$

Study of a Random Walk

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$$\frac{1}{N}Z_{Nu}(N^{\alpha_B}t) = u + \frac{1}{N}\sum_{n=0}^{\lfloor N^{\alpha_B}t\rfloor} \lambda_B\left(K_n, I_n\right)\mathrm{v}_B\left(K_n\right) \longrightarrow \mathscr{B}_u^N(t) = u + \frac{1}{N}\sum_{n=0}^{\lfloor N^2 t\rfloor} X_n.$$

Hydrodynamic limits

$$\alpha_B = \frac{5}{3} \text{ if } B \neq 0 \text{ et } \alpha_B = \frac{3}{2} \text{ if } B = 0.$$

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JKO [AAP'09] and SSS [CMP'19] proved that

$$\lim_{N\to\infty} \sum_{i=1}^{2} \int_{\mathbb{T}} \left| f\left(N^{\alpha_B} t, Nu, k, i\right) - \rho_B(t,u) \right|^2 dk = 0.$$

Introduction of a small magnetic field

Initial assumption was

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}[e(0,x)] = \int_{\mathbb{R}} J(u)\mathscr{W}_0(u)du\,.$$

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Cane [preprint] : What happens if we replace B by $B_N = BN^{-\delta}$?

An interpolation process

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$$d\nu_\delta(r) = \begin{cases} |r|^{-\frac{3}{2}-1} dr \text{ if } \delta > \frac{1}{2} \\ h_B(r)dr \text{ if } \delta = \frac{1}{2} \\ |r|^{-\frac{5}{3}-1} dr \text{ if } \delta < \frac{1}{2} \end{cases}$$

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$$\lim_{N\to\infty} N^{\alpha_\delta} \pi_{B_N}\left(\left\{(k,i), \lambda_{B_N}(k,i)\mathbf{v}_{B_N}(k) > Nr\right\}\right) = \nu_\delta(r, +\infty).$$

With

$$\alpha_\delta = \frac{5-\delta}{3} \text{ if } \delta < \frac{1}{2} \quad \text{and} \quad \alpha_\delta = \frac{3}{2} \text{ if } \delta \geq \frac{1}{2}.$$

Theorem [Cane preprint] : $N^{-1} Z_{Nu}^N(N^{\alpha_\delta} \cdot)$ converges to $Y_u^\delta(\cdot)$.

An interpolation P.D.E

$$\mathscr{D}_\delta[\phi](u) = \begin{cases} -(-\Delta)^{\frac{3}{4}}[\phi](u) \text{ if } \delta > \frac{1}{2} \\ \mathscr{D}_B[\phi] \text{ if } \delta = \frac{1}{2} \\ -(-\Delta)^{\frac{5}{6}}[\phi](u) \text{ if } \delta < \frac{1}{2} \end{cases}$$

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Let ρ_δ be the solution on $[0,T] \times \mathbb{R}$ of

$$\partial_t \rho_\delta(t,u) = \mathscr{D}_\delta[\rho_\delta](t,u),$$

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Theorem [Cane, preprint] :

$$\lim_{N\to\infty} \sum_{i=1}^{2} \int_{\mathbb{T}} \left| f^N\left(N^{\alpha_\delta}t, Nu, k, i\right) - \rho_\delta(t,u) \right|^2 dk = 0.$$

Endgame

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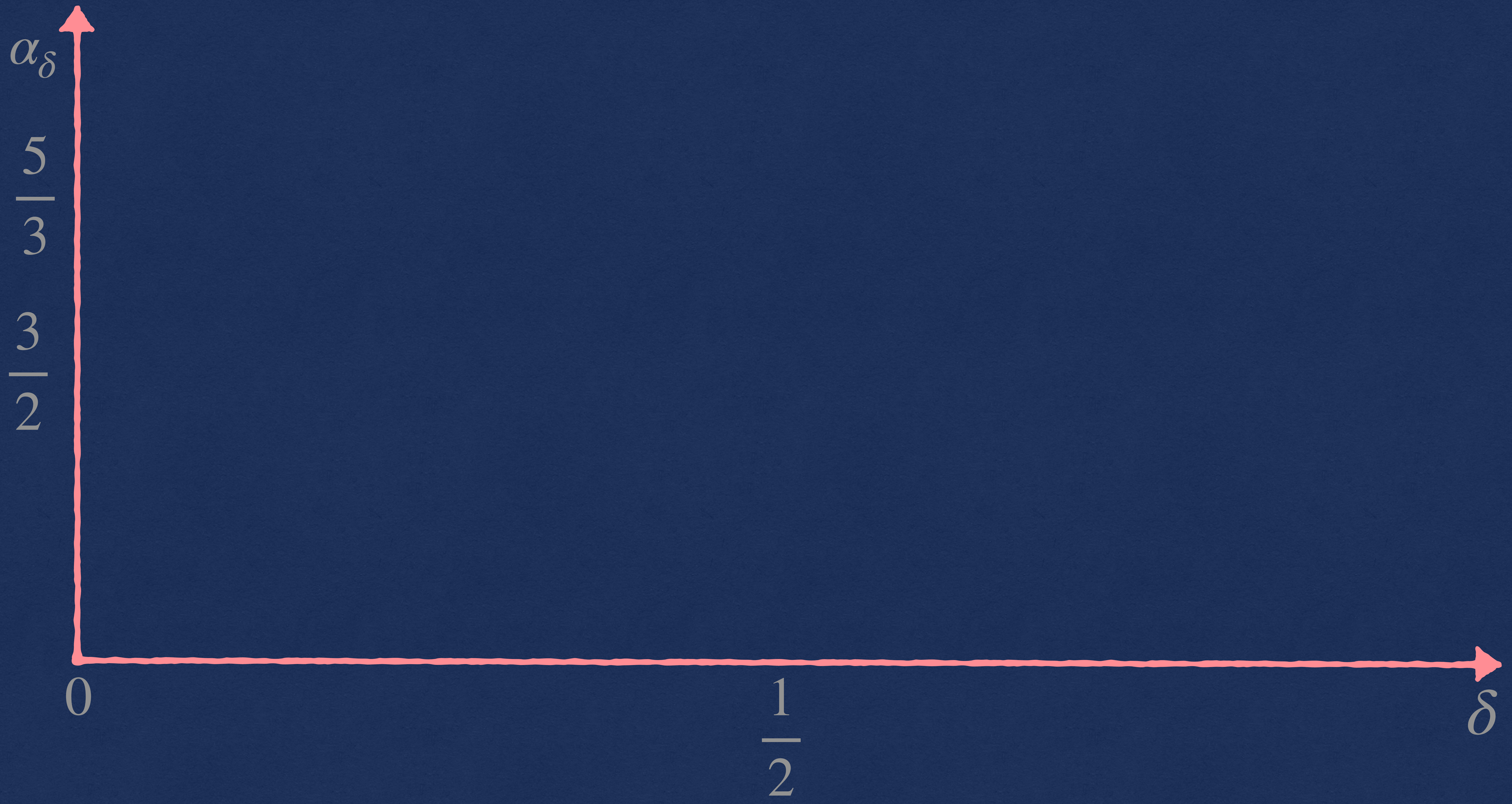
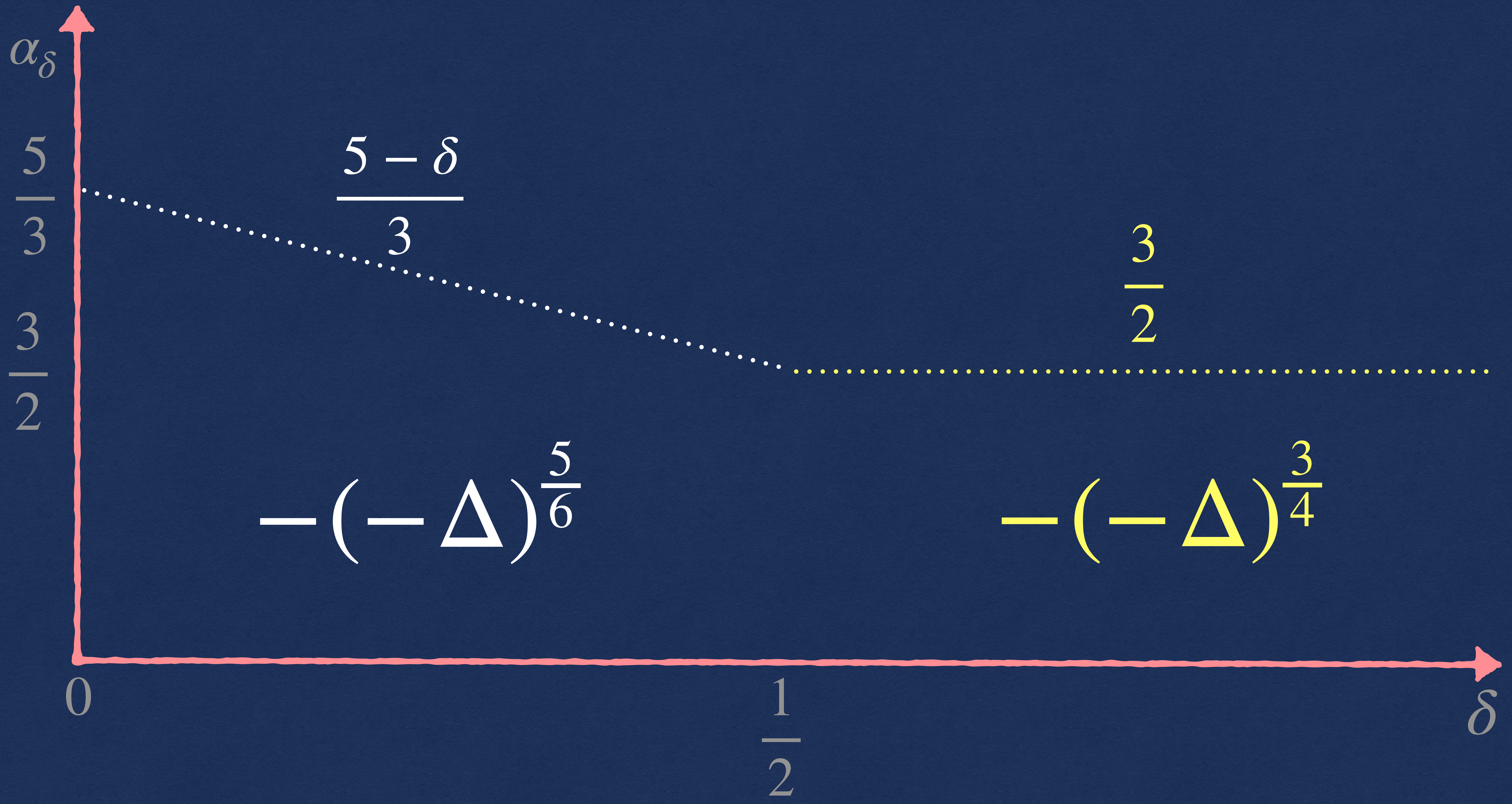


Endgame


α_δ
$\frac{5}{3}$
$\frac{3}{2}$
$\frac{5-\delta}{3}$
$\frac{3}{2}$
$-(-\Delta)^{\frac{5}{6}}$
$-(-\Delta)^{\frac{3}{4}}$
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δ

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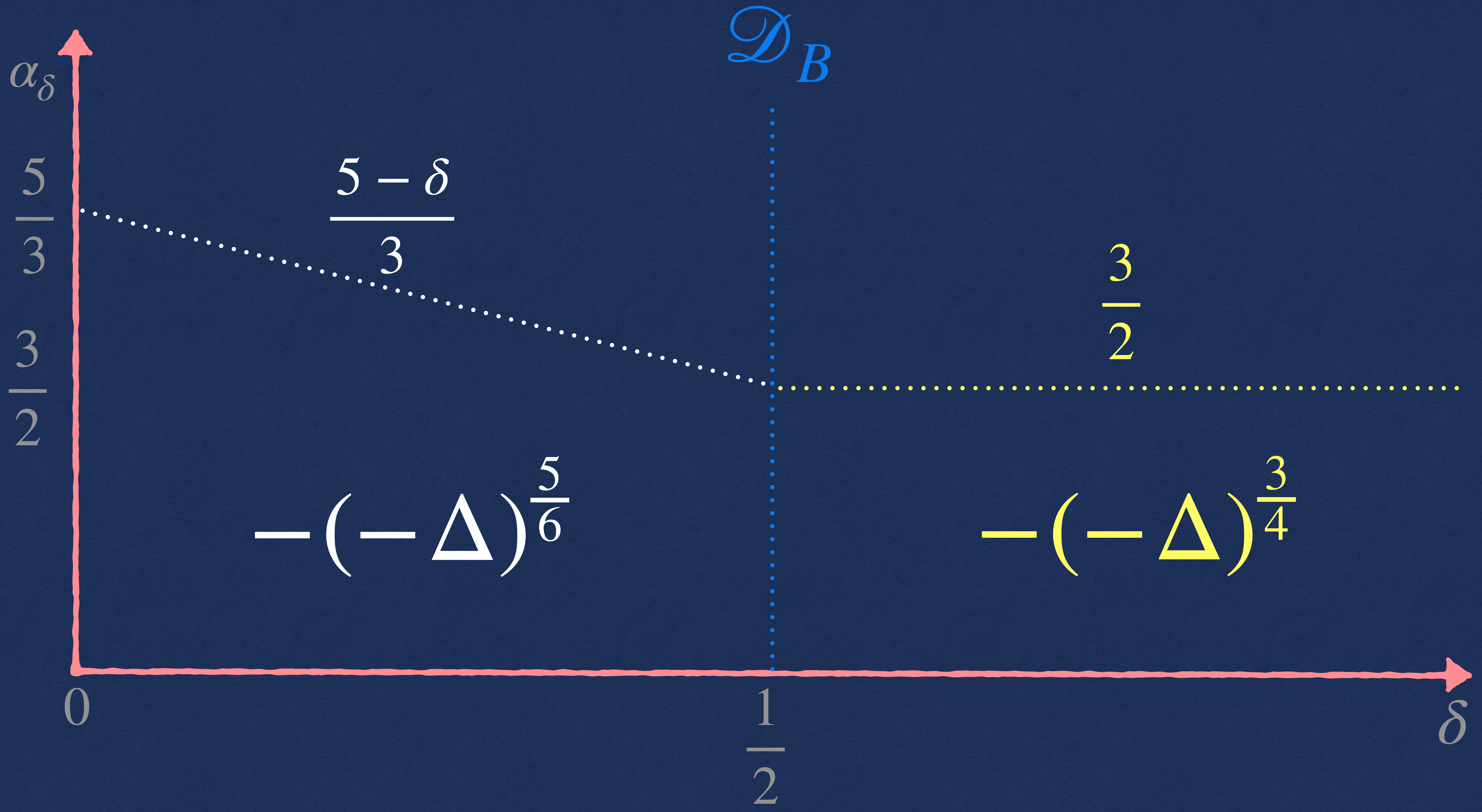

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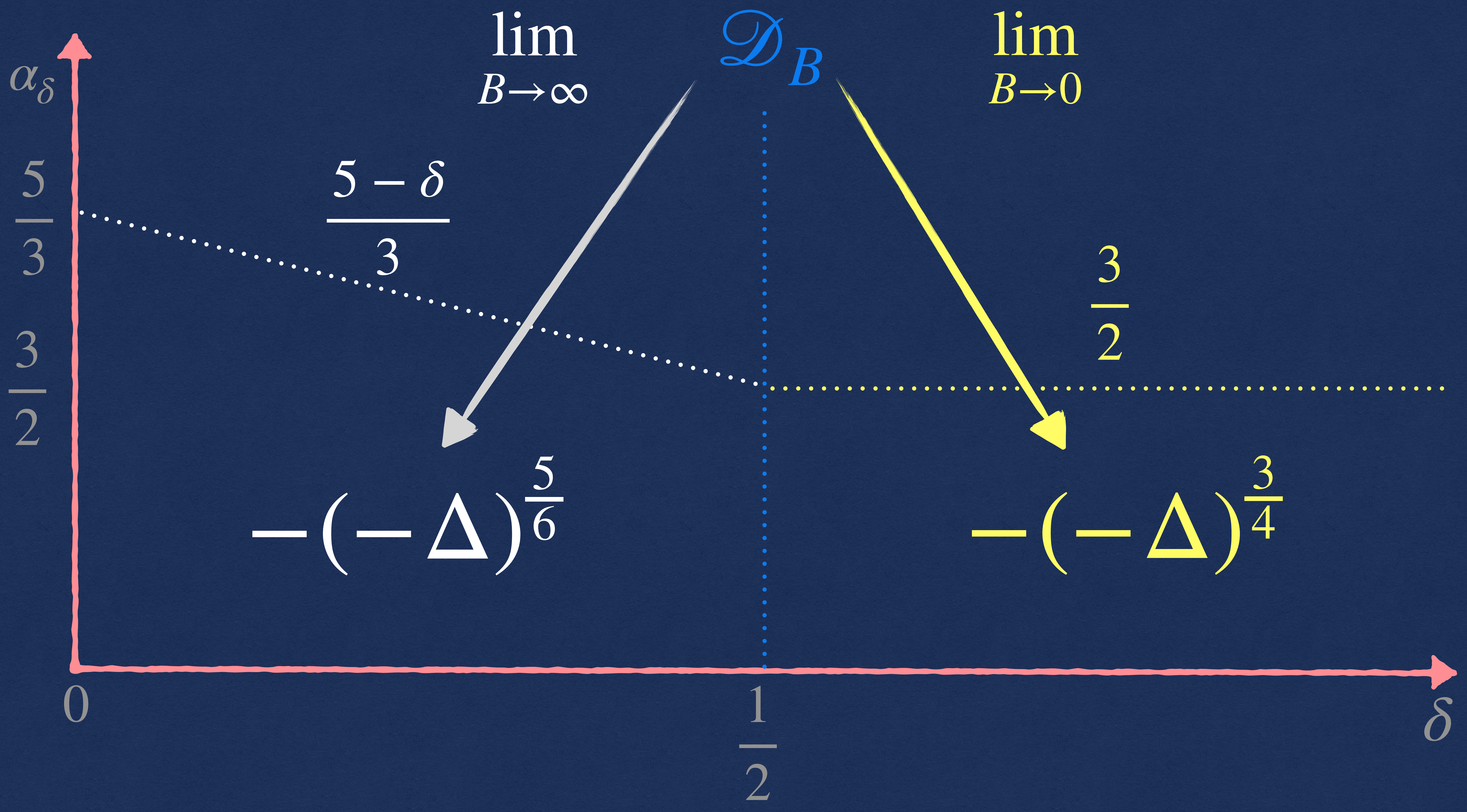

$\mathscr{D}_B$
$\lim_{B\to\infty}$
$\lim_{B\to 0}$
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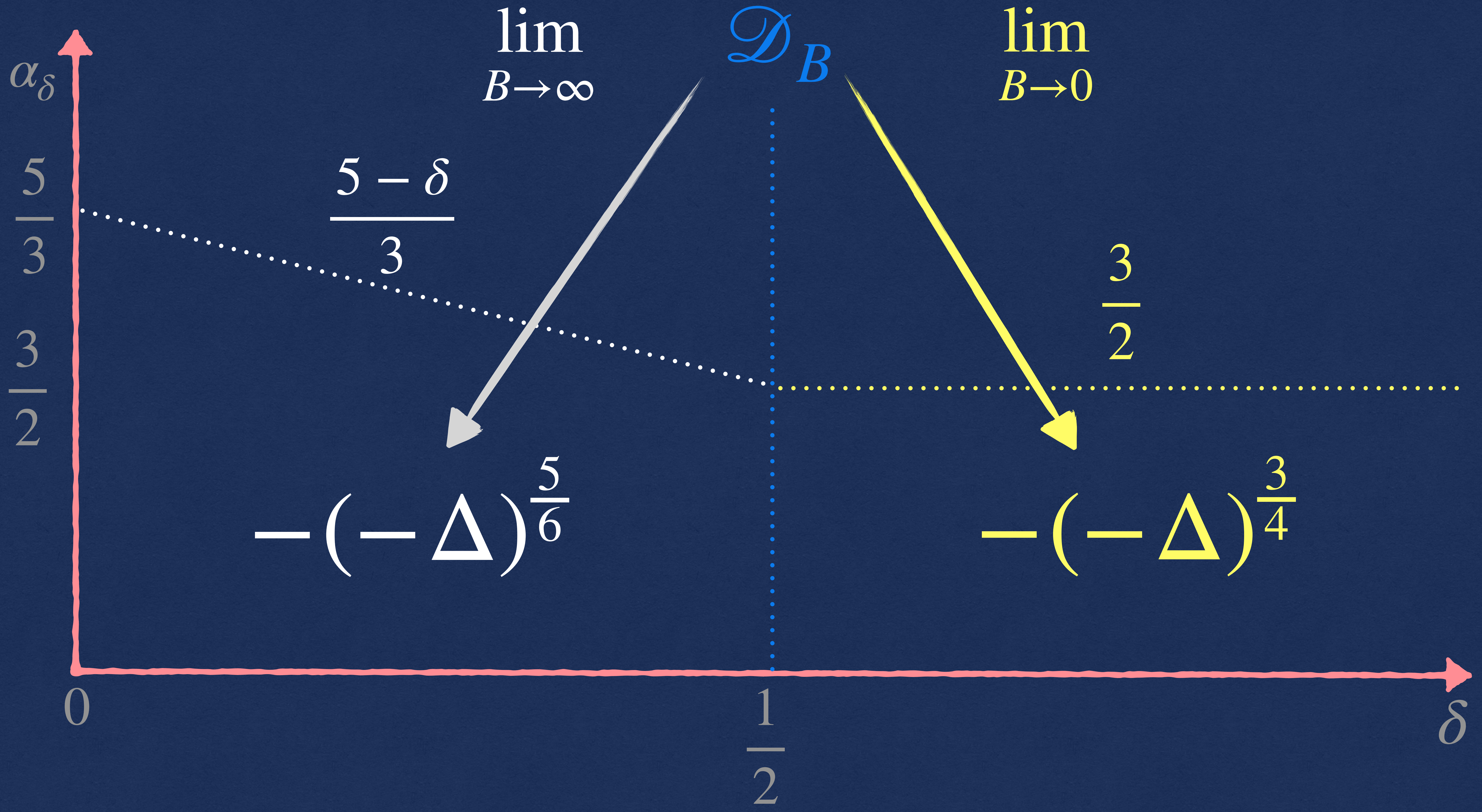
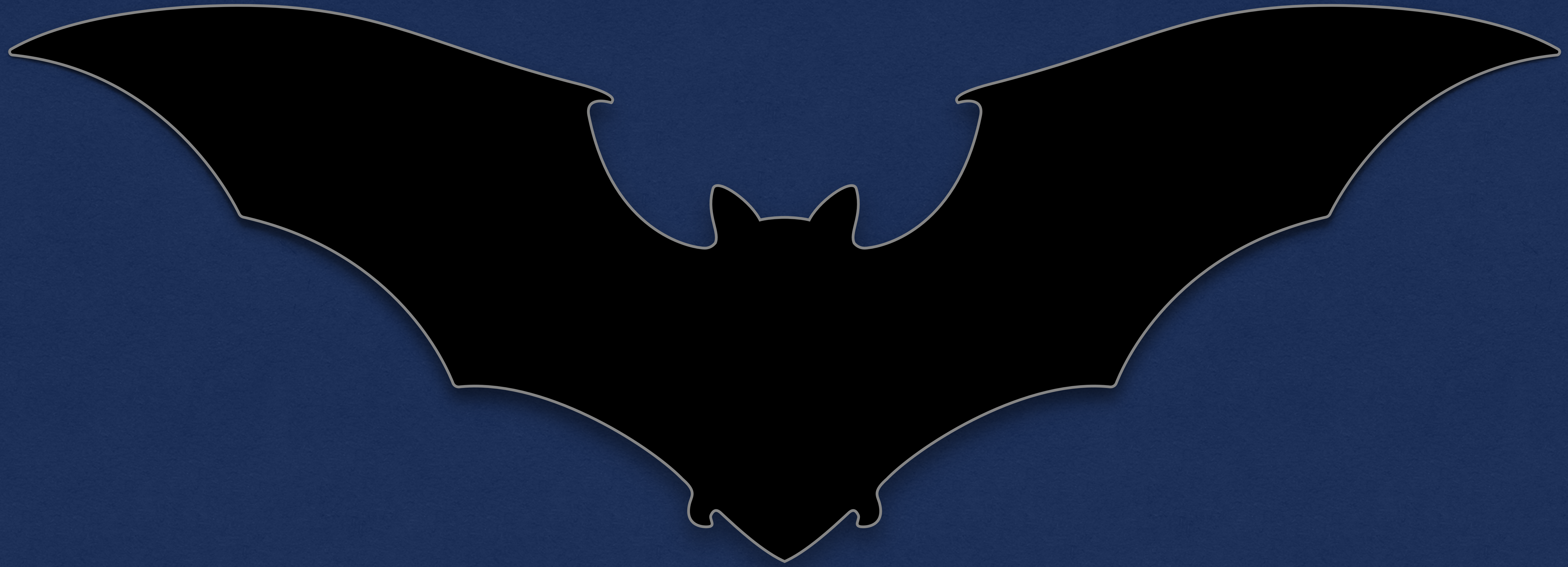


Work in progress (with Guelmame) : Study the transition in one step.



THANK YOU FOR YOUR ATTENTION