Superdiffusion transition for a noisy harmonic chain subject to a magnetic field

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Bordeaux
26/10/2022

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Introduction to the different scales

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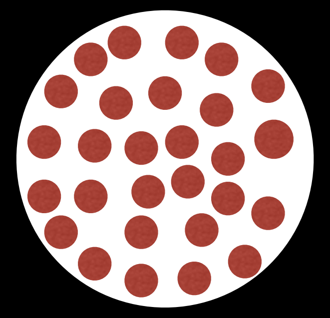
Microscopic scale

System composed of many particles and described by Newton's Laws



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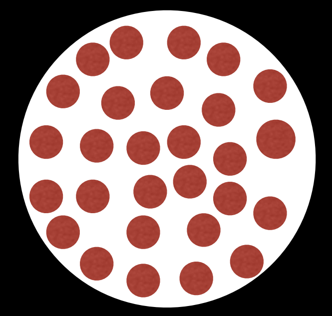

Kinetic limits →

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System described by a kinetic equation on the density f

$$\partial_t f + v\nabla_x f = Q[f,f]$$

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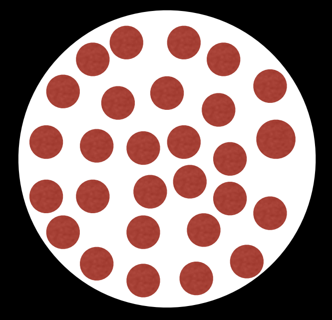

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System described by some macroscopic equations

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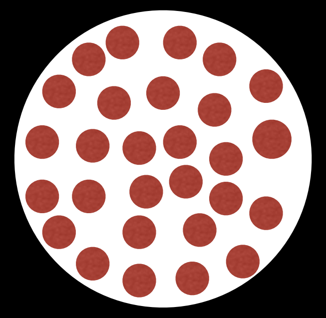

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Diffusion equation and FPUT chains

ρ a macroscopic quantity of interest $\qquad \partial_t \rho = \nabla_u \left[\kappa(\rho)\, \nabla_u \rho\right] \qquad \kappa(\rho)$ conductivity of the system

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We consider a system of interacting particles labeled by $x \in \Lambda \subset \mathbb{Z}^d$ and $d \geq 1$

$$\frac{d}{dt}q(t,x) = v(t,x)$$

$$\frac{d}{dt}v(t,x) = q(t,x+1) + q(t,x-1) - 2q(t,x) + \left(\nabla W\left(q(t,x+1) - q(t,x)\right) - \nabla W\left(q(t,x) - q(t,x-1)\right)\right)$$

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Energy : $E(t) = \frac{1}{2} \sum_{x \in \Lambda} |v(t,x)|^2 + \frac{1}{2} \sum_{x \in \Lambda} \left[\left| q(t,x+1) - q(t,x) \right|^2 + W\left(q(t,x+1) - q(t,x) \right) \right] = \sum_{x \in \Lambda} e(t,x)$

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Anharmonicity ⇄ Noise

Noisy harmonic chain submitted to a magnetic field

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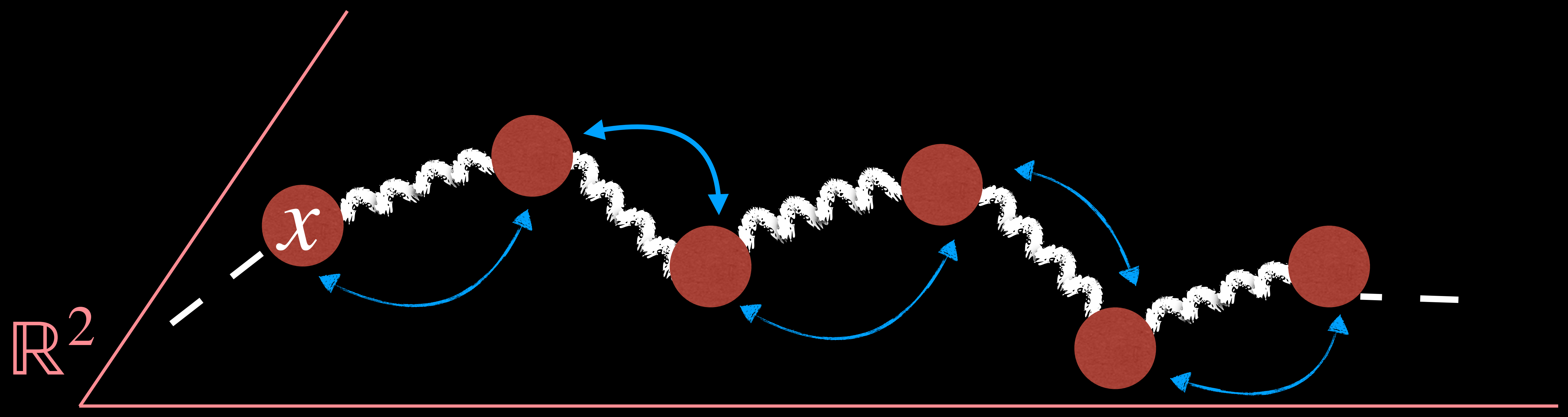

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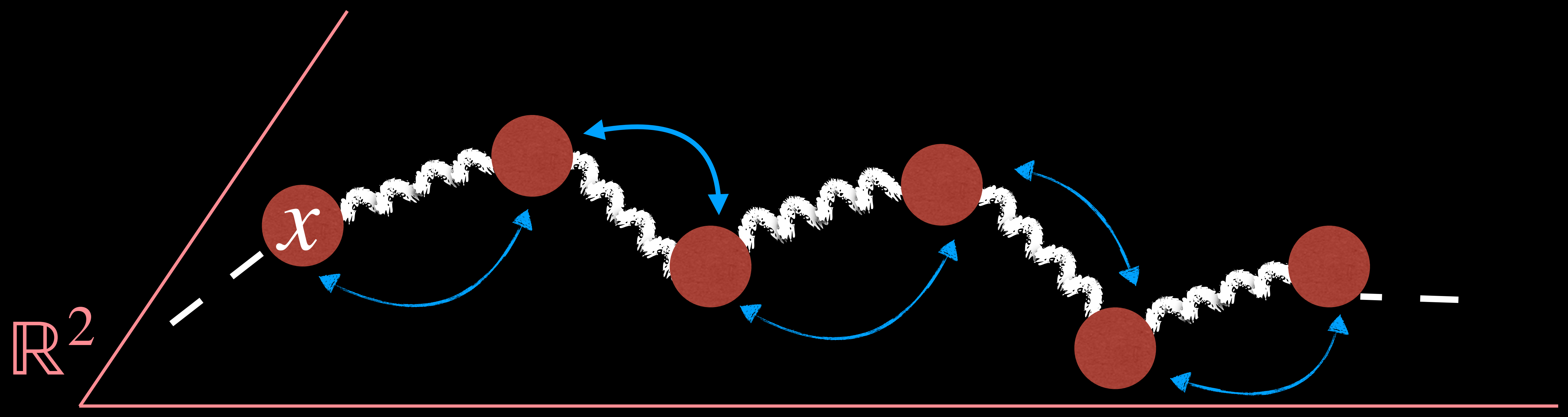

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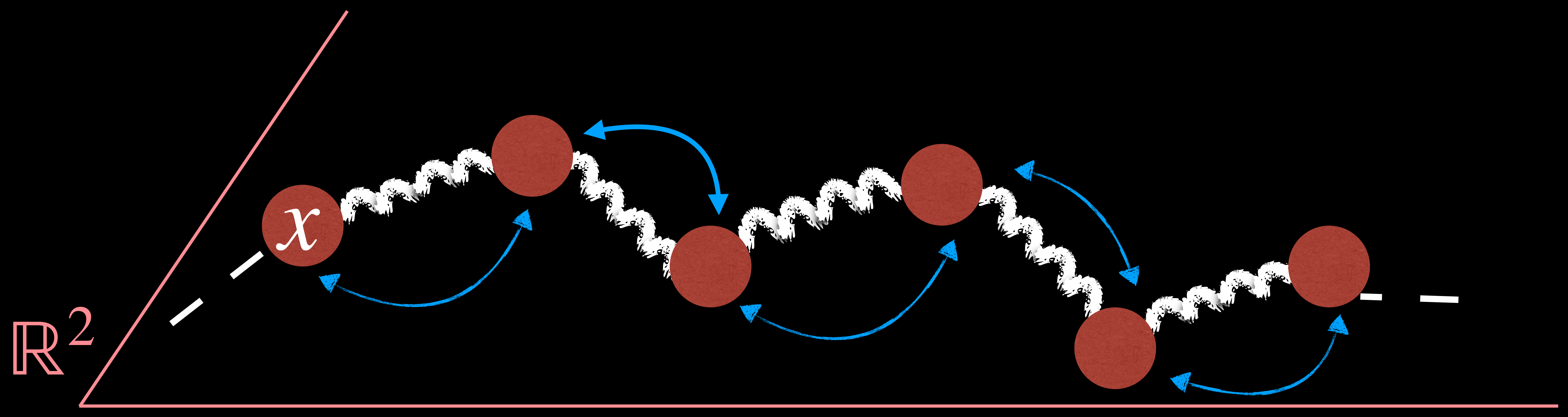

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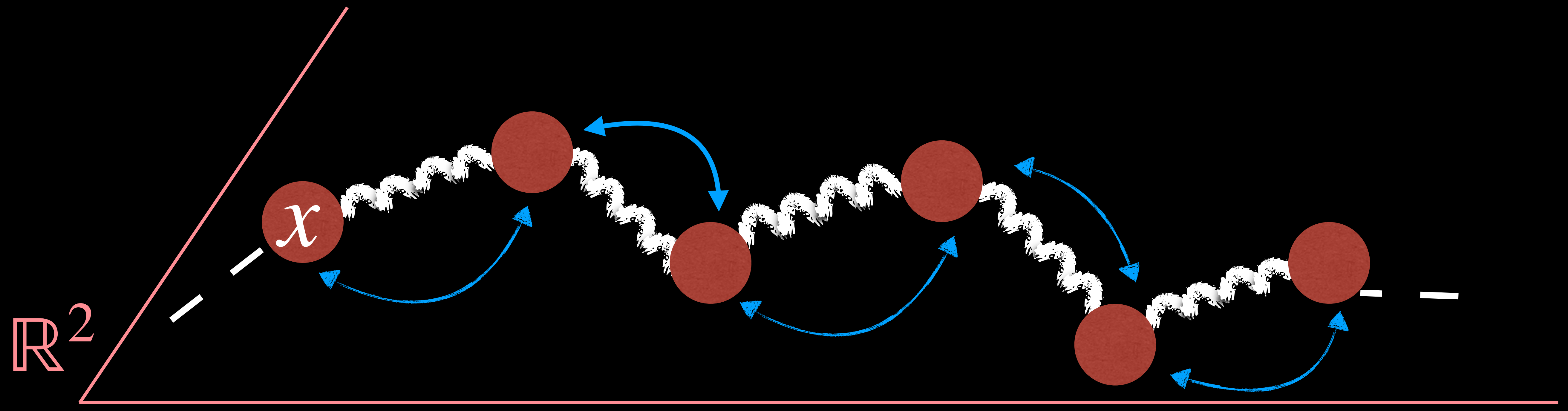

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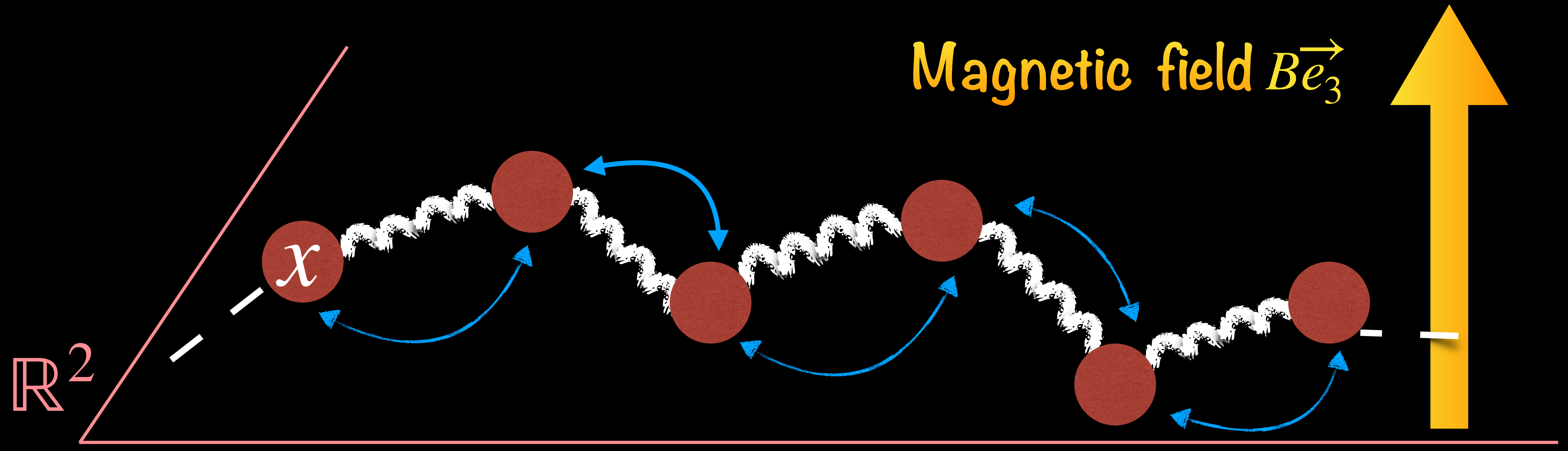

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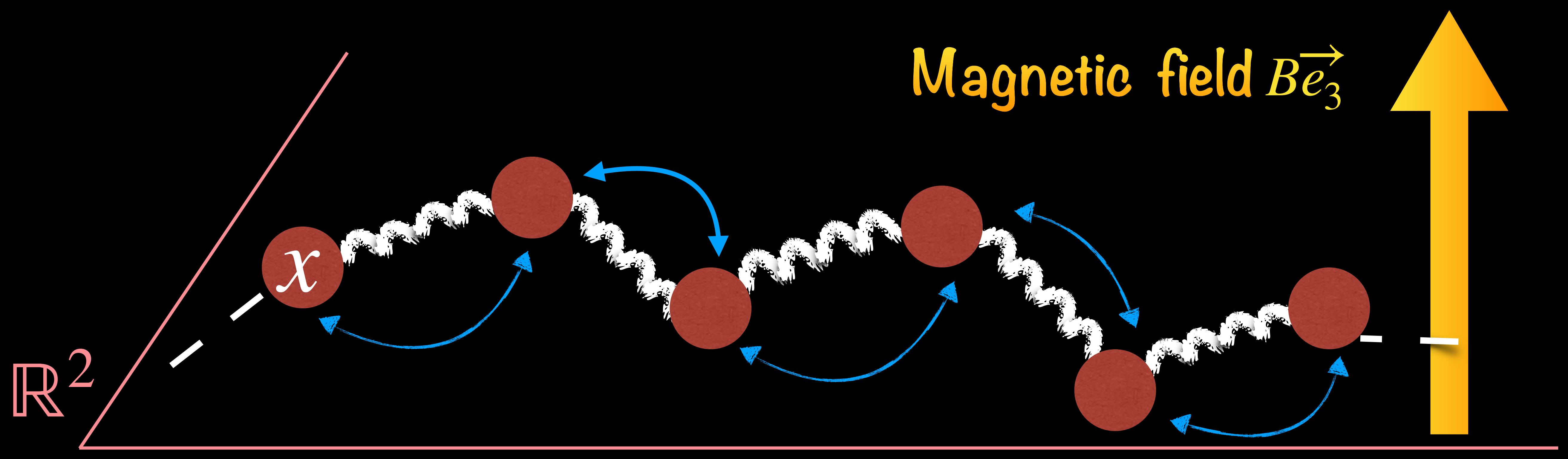

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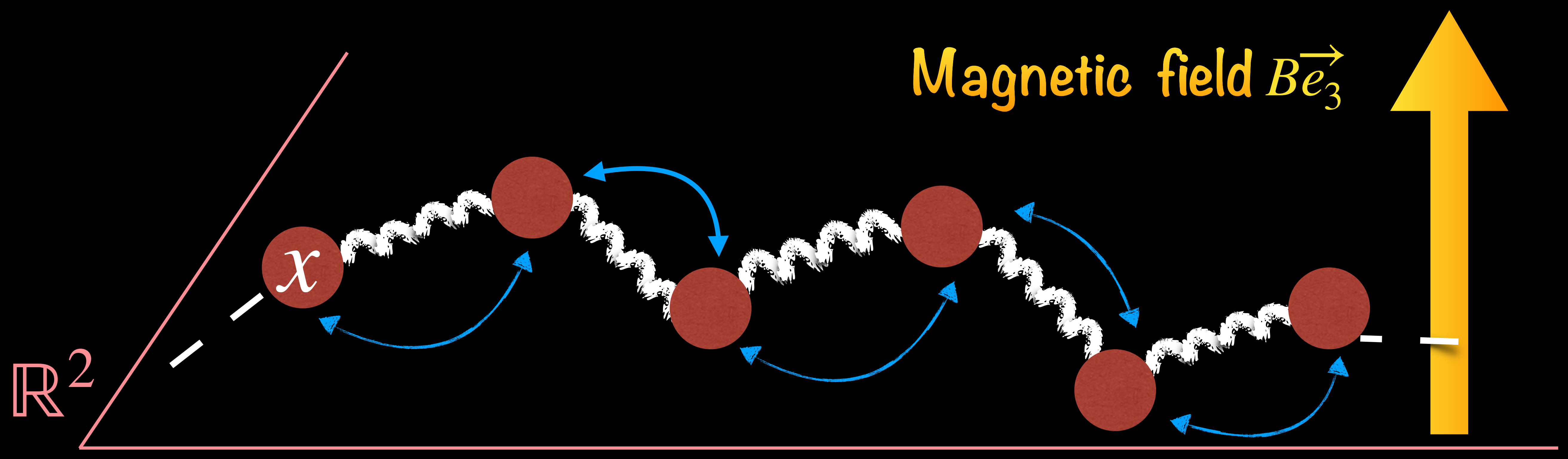

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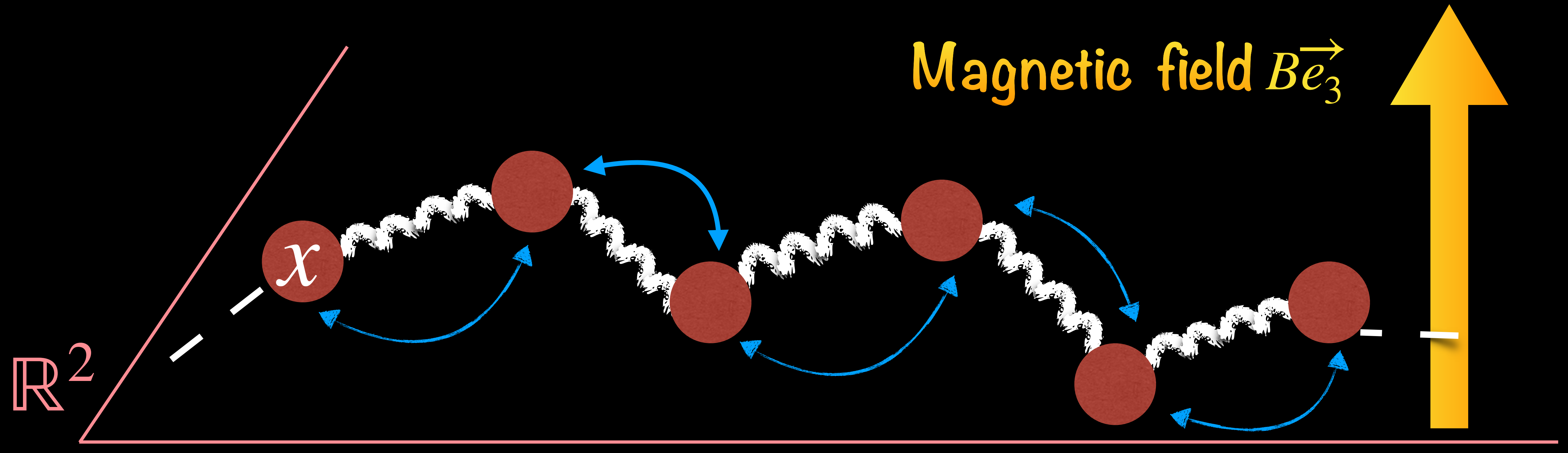

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They proved that the energy density satisfies a fractional diffusion equation with exponent $5/6$

$$\partial_t \rho_B(t,u) = -(-\Delta)^{\frac{\alpha_B}{2}} \rho_B(t,u)$$ and $$\frac{\alpha_B}{2} = \frac{5}{6} \text{ if } B \neq 0 \text{ and } \frac{\alpha_B}{2} = \frac{3}{4} \text{ if } B = 0$$

Aim of the study

Let μ^ε be the initial distribution of the system and assume that

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}\left[e(0,x)\right] = \int_{\mathbb{R}} J(u)\mathscr{W}_0(u)du$$

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$$\int_{\mathbb{T}} \left(\left|\hat{\psi}_1(t,k)\right|^2 + \left|\hat{\psi}_2(t,k)\right|^2\right) dk = E(t) = E(0)$$

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$$\langle\mathscr{W}^\varepsilon(t),J\rangle = \sum_{i=1}^{2}\int_{\mathbb{R}}dp\int_{\mathbb{T}}dk\,\mathbb{E}_{\mu_\varepsilon}\left[\hat\psi_i^*\left(t\varepsilon^{-1},k-\frac{\varepsilon p}{2}\right)\hat\psi_i\left(t\varepsilon^{-1},k+\frac{\varepsilon p}{2}\right)\right]\mathscr{F}\left[J_i\right](k,p)$$

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Macroscopic scale

Can we prove that

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}\left[e\left(t\varepsilon^{-1},x\right)\right] = \int_{\mathbb{R}} J(u)\mathscr{W}_t(u)du$$

where $\mathscr{W}_t$ solves some evolution equation?

Let $\{\hat{\psi}_{1,2}(k)\}_{k\in\mathbb{T}}$ be the eigenvectors associated to the infinitesimal generator of the deterministic dynamic

$$\int_{\mathbb{T}} \left(\left|\hat{\psi}_1(t,k)\right|^2 + \left|\hat{\psi}_2(t,k)\right|^2\right) dk = E(t) = E(0) \quad (\mathbb{T} = [0,1) \text{ is the unit torus})$$

Let $\mathscr{W}^\varepsilon = (\mathscr{W}_1^\varepsilon, \mathscr{W}_2^\varepsilon) : [0,T] \to (S\times S)'$ with $S = \{\text{smooth functions on } \mathbb{T}\times\mathbb{R}\}$. Let $J\in S\times S$ independent of k

$$\langle \mathscr{W}^\varepsilon(t), J\rangle = \varepsilon \sum_{x\in\mathbb{Z}} \mathbb{E}_{\mu^\varepsilon}\left[e\left(t\varepsilon^{-1},x\right)\right] J(\varepsilon x) + \mathcal{O}_J(\varepsilon)$$

To understand the behavior of the energy, we have to understand the one of $\mathscr{W}^\varepsilon$

Previous results on this model

BOS [ARMA'10] and SSS [CMP'19] proved that $\mathscr{W}^\varepsilon$ converges to f_B where

$$\partial_t f_B(t,u,k,i) + \frac{\mathbf{v}_B(k)}{2\pi}\partial_u f_B(t,u,k,i) = \mathscr{L}_B[f_B](t,u,k,i)$$

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Here

$$\mathscr{L}_B\left[f_B\right](t,u,k,i) = \sum_{j=1}^{2}\int_{\mathbb{T}} \theta_{i,B}^2(k) R\left(k,k'\right)\theta_{j,B}^2\left(k'\right)\left(f_B\left(t,u,k',j\right) - f_B(t,u,k,i)\right) dk'$$

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With

$$\mathbf{v}_B(k) = \frac{\sin(\pi k)\cos(\pi k)}{\sqrt{\sin^2(\pi k) + \frac{B^2}{4}}}$$

$$\theta_{1/2,B}^2 = \frac{1}{2} \pm \frac{B}{4\sqrt{\sin^2(\pi k) + \frac{B^2}{4}}}$$

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We want to study the long time behavior of f_B

Brownian motion and diffusion equation

Let $(X_n)_{n\in\mathbb{N}^*}$ a sequence of **i.i.d** random variables such that $\mathbb{P}\left(X_1 = 1\right) = \mathbb{P}\left(X_1 = -1\right) = \dfrac{1}{2}$

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Let u in $\mathbb{R}$ and $t \geq 0$ we define

$$\mathcal{B}_u^N(t) = u + \frac{1}{N}\sum_{n=1}^{\lfloor N^2 t\rfloor} X_n$$

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ρ is solution of $\partial_t \rho(t,u) = \frac{1}{2}\Delta[\rho](t,u)$

Brownian motion induces diffusion

Lévy process and fractionnal diffusion equation

Let $\alpha \in (1,2)$ and σ a measure on $\mathbb{R}^*$ such that $d\sigma(r) = |r|^{-\alpha-1}dr$ then

$$\int_{\mathbb{R}^*} \min(1,r^2)d\sigma(r) < +\infty \text{ and } \int_{\mathbb{R}^*} r^2 d\sigma(r) = +\infty$$

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$$Y_u^N(t) = u + \frac{1}{N}\sum_{n=1}^{\lfloor N^\alpha t\rfloor} X_n \longrightarrow \mathscr{B}_u^N(t) = u + \frac{1}{N}\sum_{n=1}^{\lfloor N^2 t\rfloor} X_n$$

Shorter time scale

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Shorter time scale

$Y_u^N(\cdot) \to Y_u(\cdot)$ where $Y_u(\cdot)$ is a Lévy process on $\mathbb{R}$ starting from u with measure σ

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We define

$$\rho(t,u) = \mathbb{E}\left[\rho_0\left(Y_u(t)\right)\right] \longrightarrow \rho(t,u) = \mathbb{E}\left[\rho_0\left(\mathscr{B}_u(t)\right)\right]$$

Then

$$\partial_t \rho(t,u) = -(-\Delta)^{\frac{\alpha}{2}}[\rho](t,u) \longrightarrow \partial_t \rho(t,u) = \frac{1}{2}\Delta[\rho](t,u)$$

Lévy process induces fractionnal diffusion.

A jump process

We want to study the long time behavior of f_B where $$\partial_t f_B(t,u,k,i) + \frac{\mathbf{v}_B(k)}{2\pi}\partial_u f_B(t,u,k,i) = \mathscr{L}_B[f_B](t,u,k,i)$$

$$\mathscr{L}_B\left[f_B\right](t,u,k,i) = \sum_{j=1}^{2}\int_{\mathbb{T}} \theta_{i,B}^2(k) R\left(k,k'\right)\theta_{j,B}^2\left(k'\right)\left(f_B\left(t,u,k',j\right) - f_B(t,u,k,i)\right) dk'$$

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$\mathscr{L}_B$ is the infinitesimal generator of a jump process $\left(K^B(\,\cdot\,), I^B(\,\cdot\,)\right) \in \left(\mathbb{T}\times\{1,2\}\right)$

- $\left(K^B(0), I^B(0)\right) = (k,i)$
- The process waits a time $\lambda_B(k,i)$
- The process goes from (k,i) to (k',j) with probability $P_B(k,i,dk',j)$

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Then

$$f_B(t,u,k,i) = \mathbb{E}\left[f^0\left(Z_u^B(t), K_k^B(t), I_i^B(t)\right)\right] \longrightarrow f(t,u) = \mathbb{E}\left[f_0\left(\mathscr{B}_u(t)\right)\right]$$

Study of a Random Walk

Let $\mathcal{N}(t)$ be the number of jumps until time t

$$Z_u^B(t) = u - \frac{1}{2\pi}\int_0^t \mathbf{v}_B\left(K^B(s)\right) ds = u - \sum_{n=1}^{\mathcal{N}(t)} \tau_{n-1}\, \lambda_B\left(K_{n-1}^B, I_{n-1}^B\right) \frac{\mathbf{v}_B\left(K_{n-1}^B\right)}{2\pi} \longrightarrow S_N = \sum_{n=1}^{\mathcal{N}(t)} X_n$$

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Let $Y_u^B(\,\cdot\,)$ be the Lévy process with Lévy measure ν_B

$$d\nu_B(r) = \begin{cases} |r|^{-\frac{3}{2}-1} dr \text{ if } B = 0 & \longleftrightarrow -(-\Delta)^{\frac{3}{4}} \\ |r|^{-\frac{5}{3}-1} dr \text{ if } B \neq 0 & \longleftrightarrow -(-\Delta)^{\frac{5}{6}} \end{cases}$$

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Let π_B the invariant measure of $\left(K_n^B, I_n^B\right)_{n\in\mathbb{N}}$ and $r > 0$ then

$$\lim_{N\to\infty} N^{\alpha_B}\pi_B\left(\left\{(k,i), \lambda_B(k,i)\mathbf{v}_B(k) > Nr\right\}\right) = \nu_B(r,+\infty) = |r|^{-\alpha_B}$$ with $\alpha_B = \frac{5}{3}$ if $B \neq 0$ and $\alpha_B = \frac{3}{2}$ if $B = 0$

Study of a Random Walk

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Let $Y_u^B(\cdot)$ be the Lévy process with Lévy measure ν_B

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Study of a Random Walk

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Macroscopic scale

where

$$\partial_t \rho_B(t,u) = -(-\Delta)^{\frac{\alpha_B}{2}}\rho_B(t,u)$$

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$$\frac{\alpha_B}{2} = \frac{5}{6} \text{ if } B\neq 0 \text{ and } \frac{\alpha_B}{2} = \frac{3}{4} \text{ if } B=0$$

Let's sum up

Initial assumption was

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}[e(0,x)] = \int_{\mathbb{R}} J(u)\mathscr{W}_0(u)du\,.$$

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Cane [submitted] : **What happens if we replace B by $B_N = BN^{-\delta}$?**

- $\delta = 0$ constant magnetic field $\longleftrightarrow$ $-(-\Delta)^{\frac{5}{6}}$
- $\delta = \infty$ no magnetic field $\longleftrightarrow$ $-(-\Delta)^{\frac{3}{4}}$

Introduction of a small magnetic field

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Theorem [Cane]: $N^{-1}Z_{Nu}^{B_N}\left(N^{\alpha_\delta}\cdot\right)$ converges to the Lévy process $Y_u^\delta(\cdot)$ with Lévy measure ν_δ where

$$\alpha_\delta = \frac{5-\delta}{3} \text{ if } \delta < \frac{1}{2} \text{ and } \alpha_\delta = \frac{3}{2} \text{ if } \delta \geq \frac{1}{2}$$

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Let C_δ be a positive constant and ρ_δ be the solution on $[0,T] \times \mathbb{R}$ of

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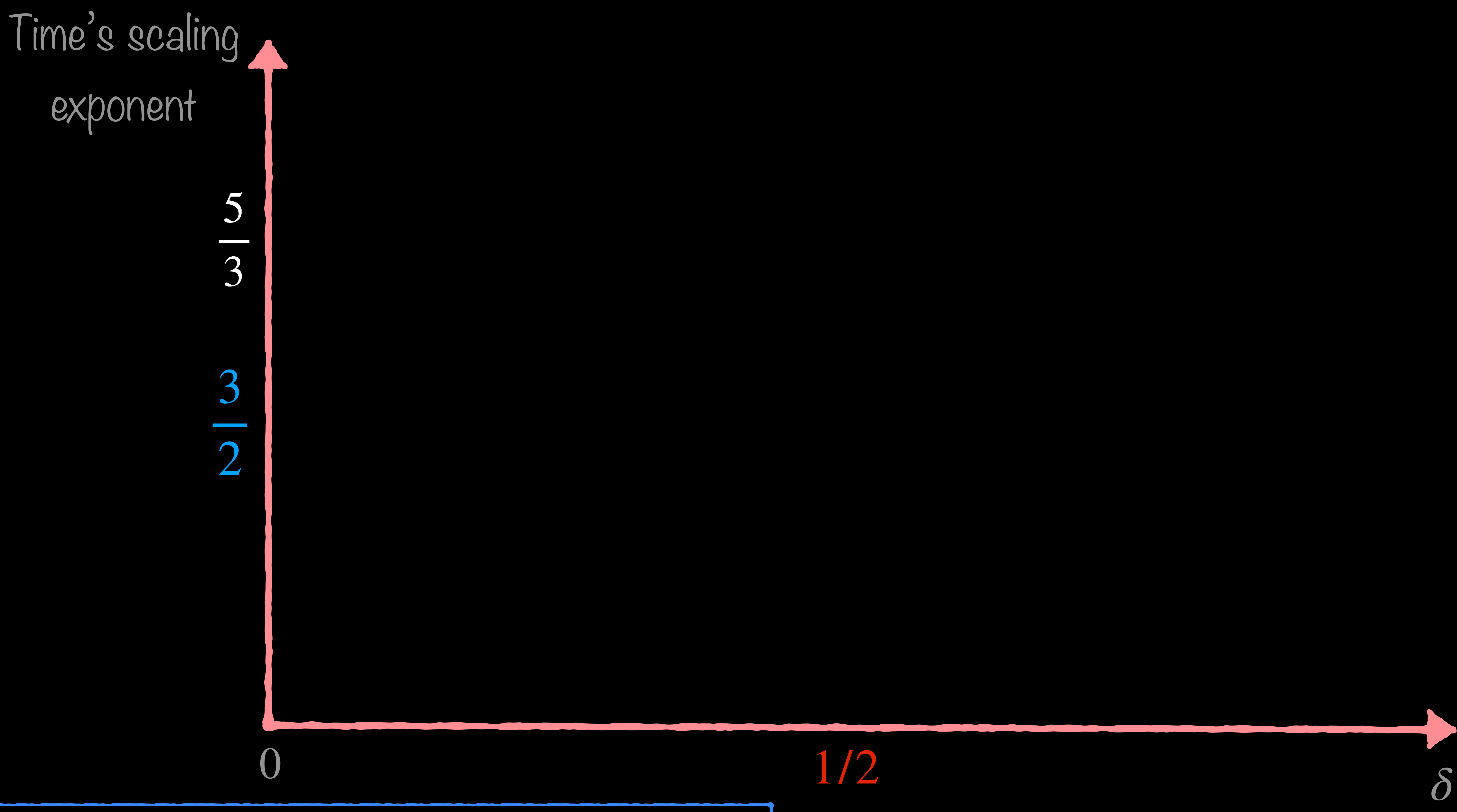

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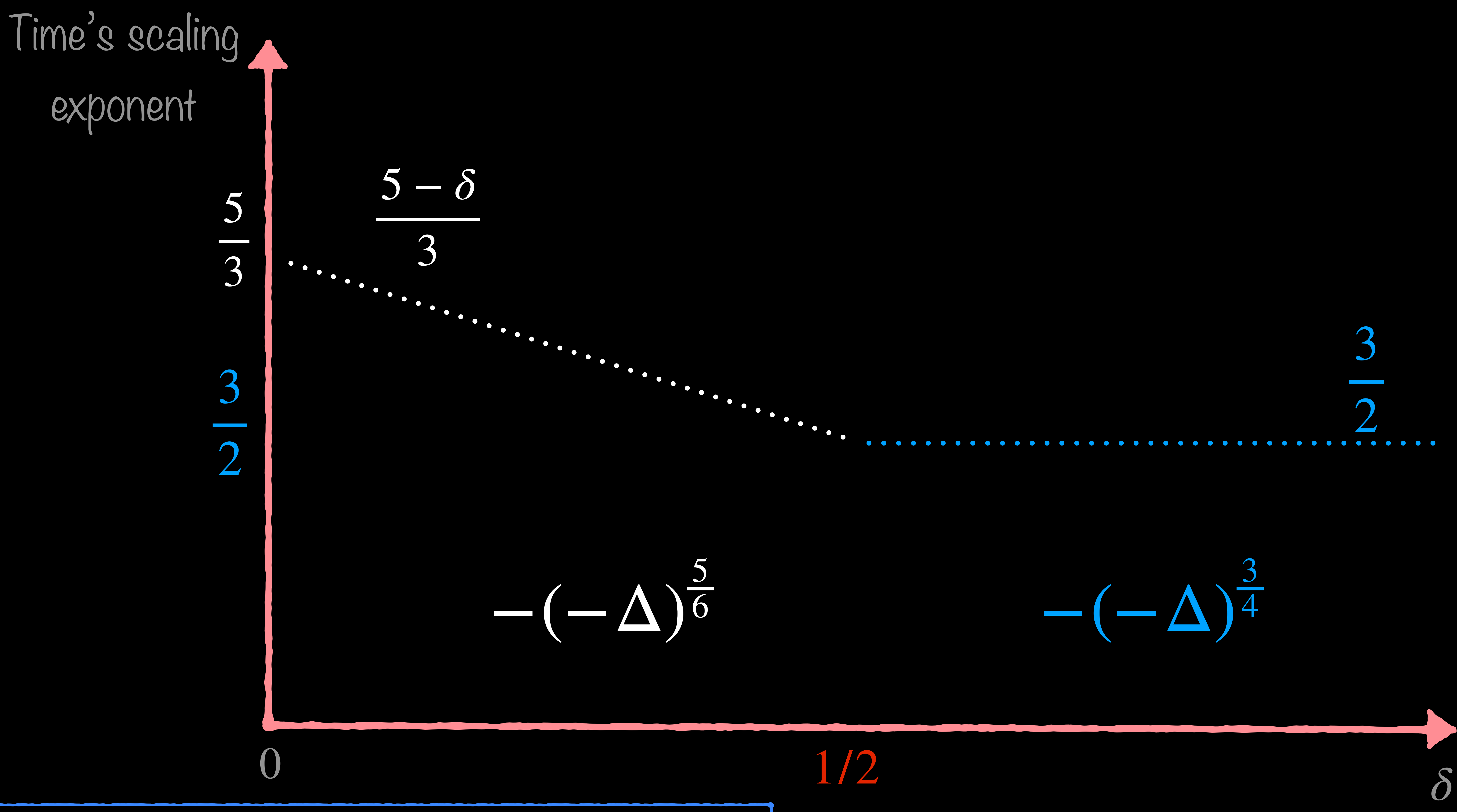

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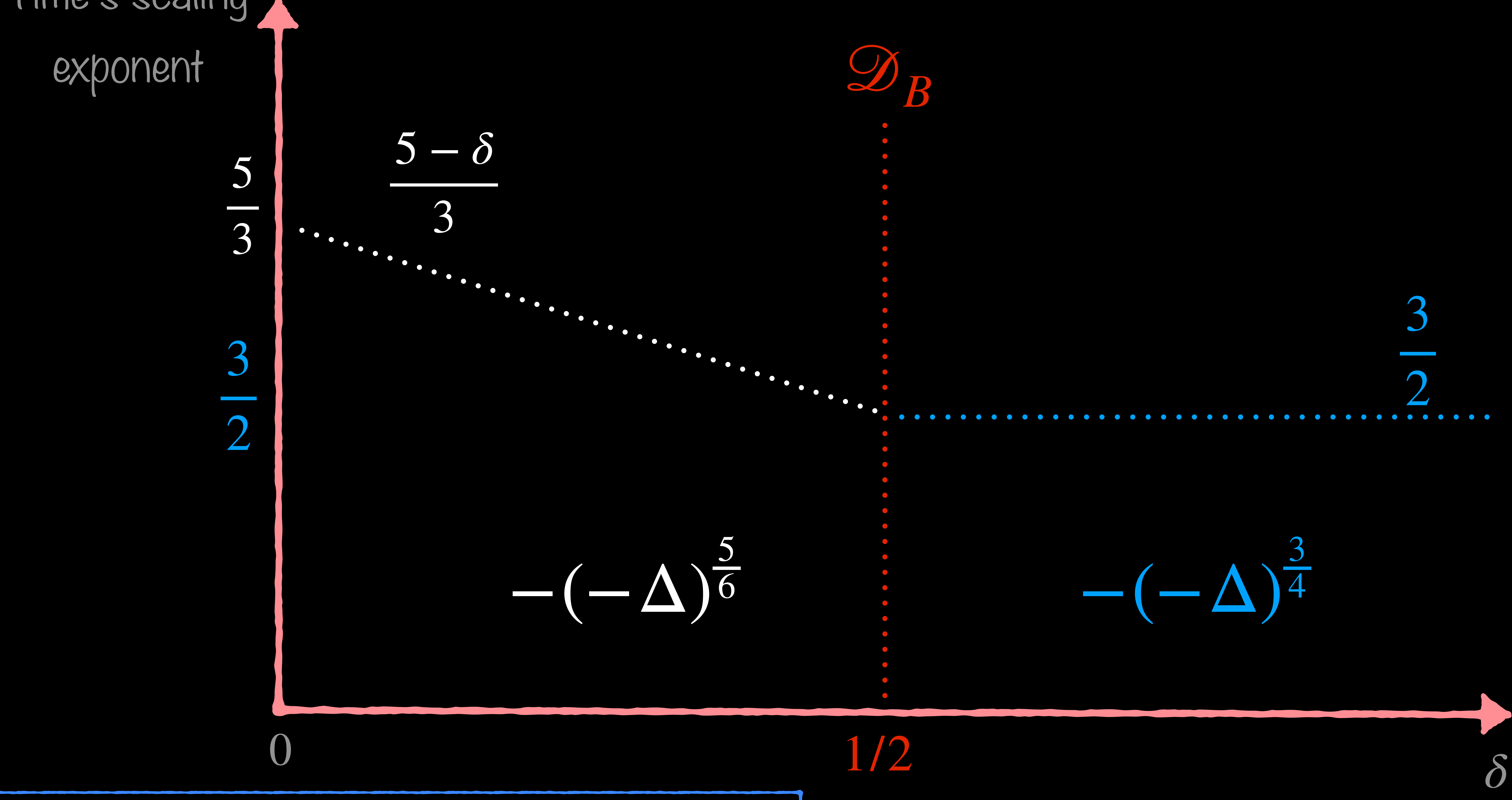

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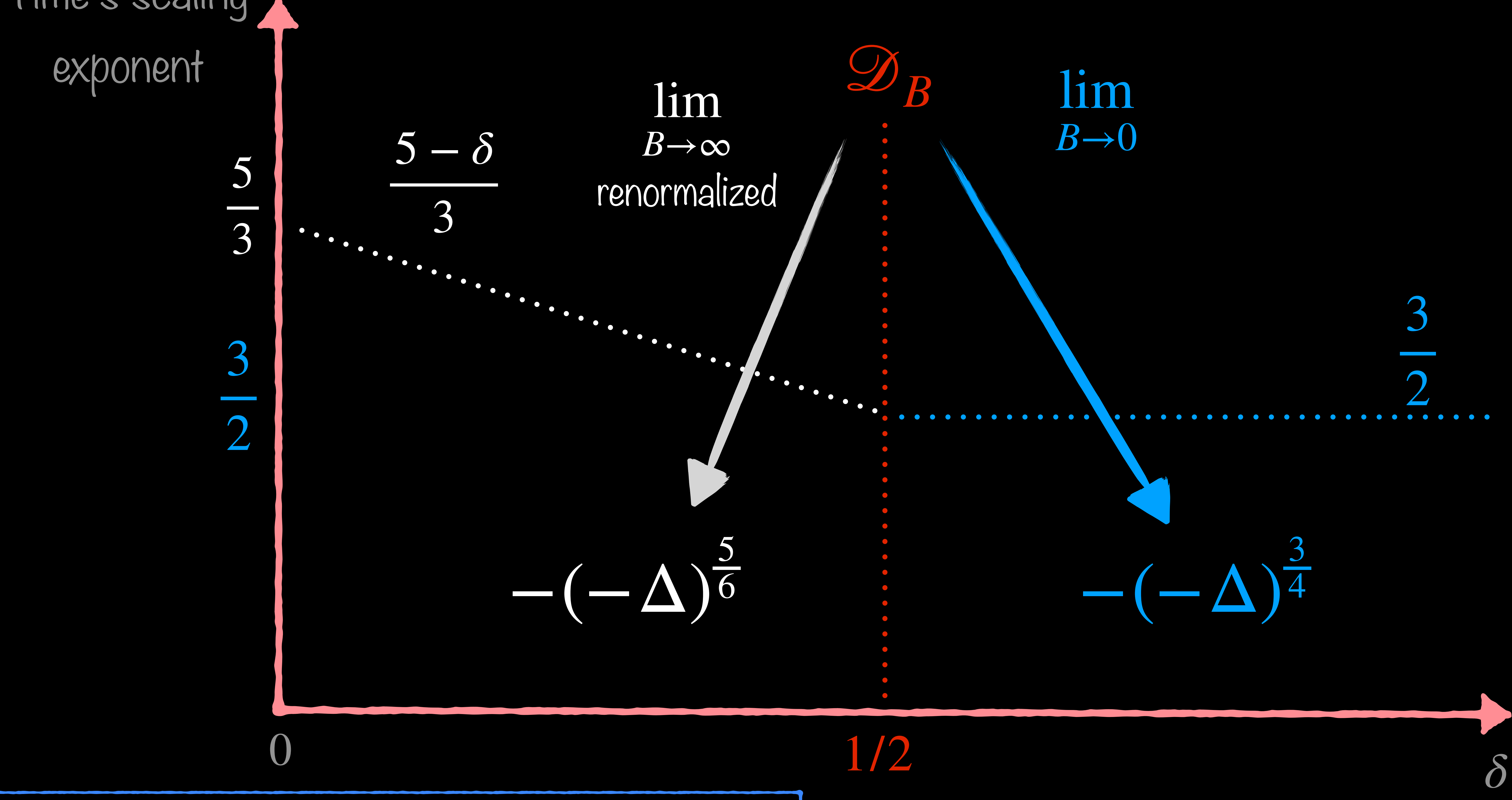

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Summary and open problems

Transition between two fractional diffusion equations depending on the intensity of the magnetic field [C submitted]

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Transition between two fractional diffusion equations depending on the intensity of the magnetic field [C submitted]

Some open problems

- Transition in one step (work in progress with **Guelmame**)
- Transition between two fractional Laplacian with a magnetic interface (work in progress with **Simon**)

THANK YOU FOR YOUR ATTENTION

Interpolation operator

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For each $\xi \in \mathbb{R}$

$$\Phi_\delta(\xi) = \int_{\mathbb{R}} \left(1 - \exp(\mathbf{i}\xi r) + \mathbf{i}\xi r 1_{|r|\leq 1}\right) d\nu_\delta(r)$$

For each smooth functions ϕ

$$\mathfrak{L}_\delta\left[\phi\right] = \int_{\mathbb{R}} \mathscr{F}\left[\phi\right]\left(\xi\right) \Phi_\delta(\xi) \exp(2\mathbf{i}\pi p \xi) d\xi$$