Superdiffusion transition for a noisy harmonic chain subject to a magnetic field

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FPUT chains

We consider a system of N interacting particles labelled by x with mass equal to one .

$$x \in \{1, \ldots, N\} \qquad \frac{d}{dt} q(t,x) = v(t,x)$$

$$\frac{d}{dt} v(t,x) = q(t,x+1) + q(t,x-1) - 2q(t,x) + \alpha \Big(W'\big(q(t,x+1) - q(t,x)\big) - W'\big(q(t,x) - q(t,x-1)\big) \Big) .$$

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$$E(t) = \frac{1}{2} \sum_{x=1}^{N} |v(t,x)|^2 + \frac{1}{2} \sum_{x=1}^{N} \big(q(t,x) - q(t,x-1)\big)^2 + \alpha W\big(q(t,x) - q(t,x-1)\big) = \sum_{x=1}^{N} e(t,x)$$

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$$\partial_t S(t,x) = \nabla_x\big(\kappa(S)\,\nabla_x S(t,x)\big)$$

κ is the conductivity of the system

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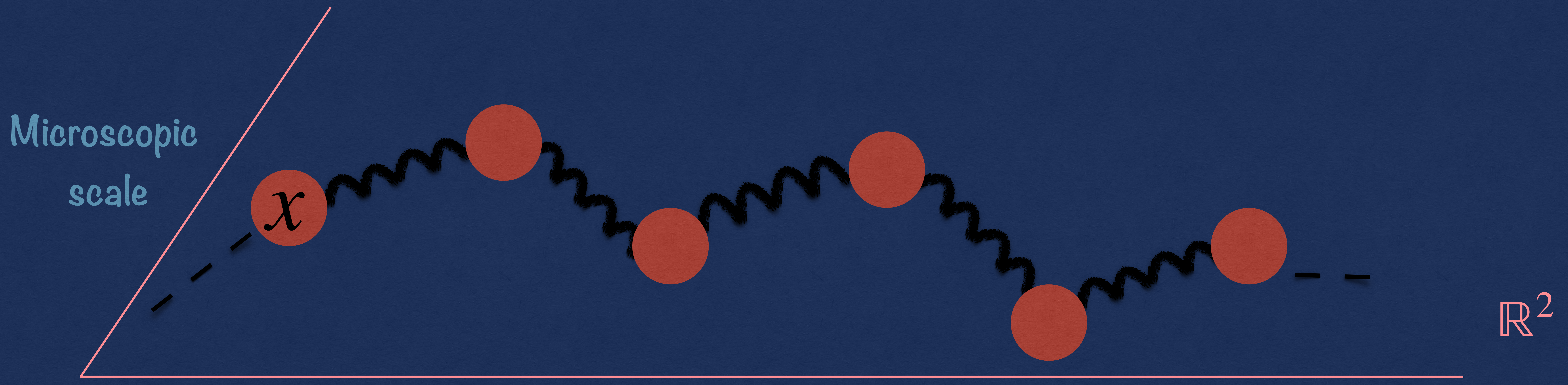
In one dimensional systems heat conductivity seems to be anomalous for $\alpha \neq 0,\ \kappa \sim N^{\delta}$

Harmonic chain submitted to a momentum conserving noise



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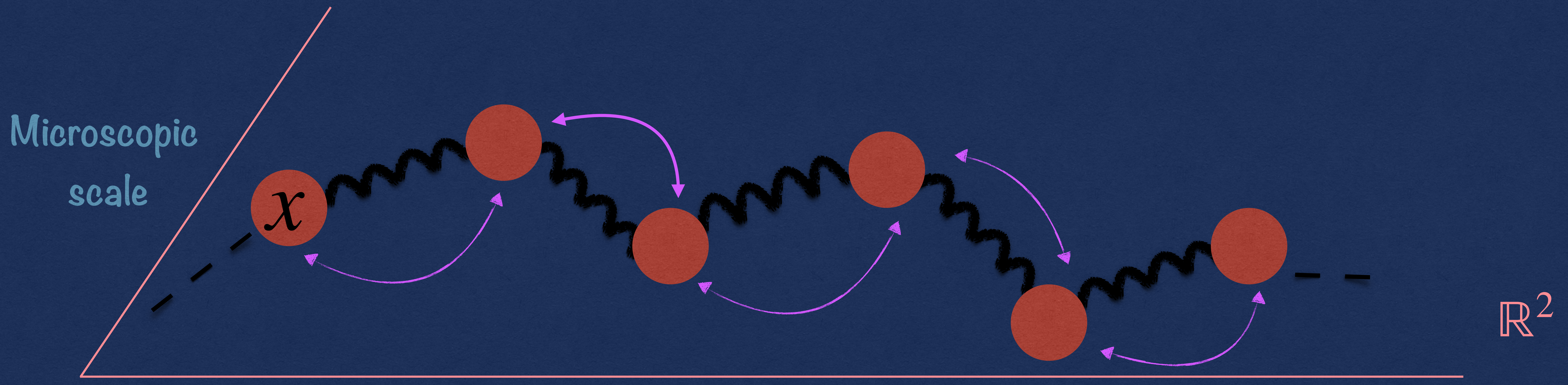

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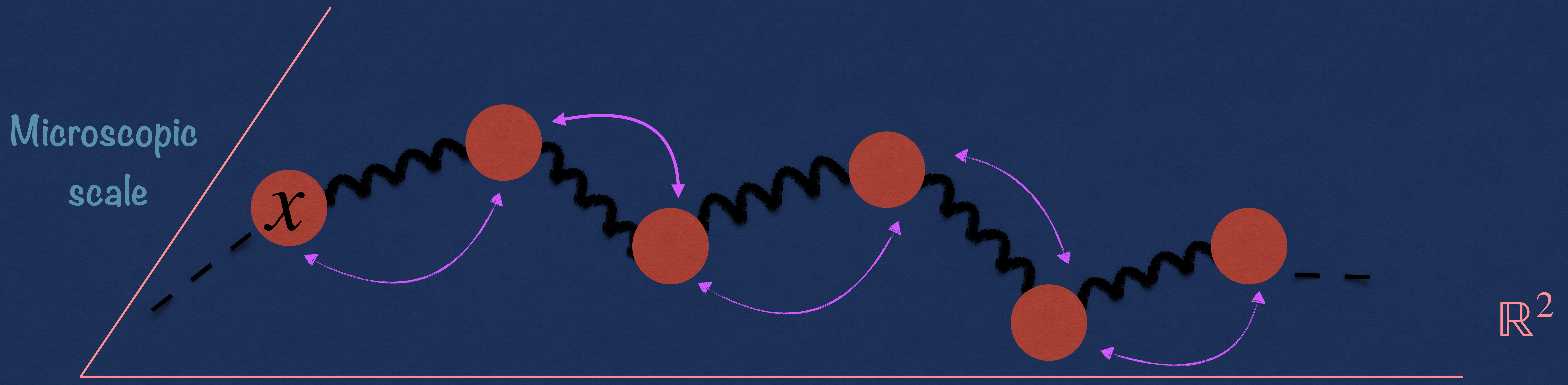

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The noise preserves the energy and the momentum .

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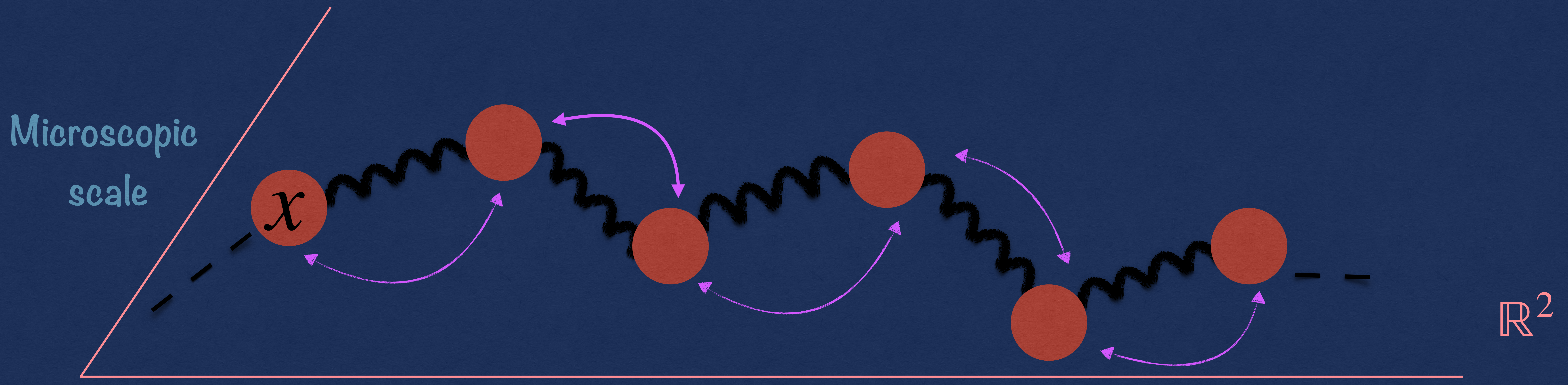

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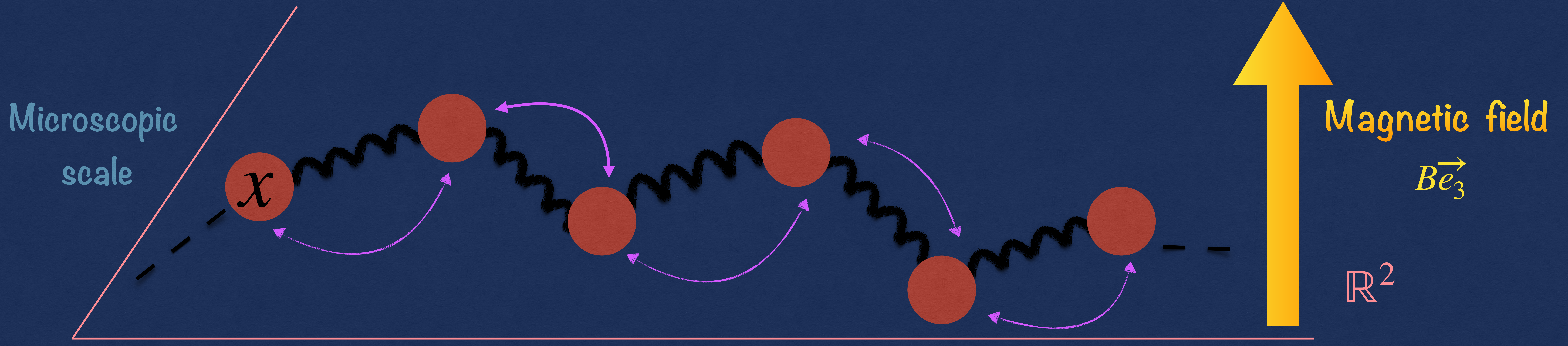

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- BOS [ARMA'10] study this system when ε goes to zero.
- SSS [CMP' 19] add a magnetic field of intensity B to the deterministic system.

Aim of the study

Let μ^ε be the initial distribution of the system. $\varepsilon \to 0$

Natural assumption :

$$\lim_{\varepsilon\to 0} \varepsilon \sum_{x\in\mathbb{Z}} J(\varepsilon x)\mathbb{E}_{\mu^\varepsilon}[e(0,x)] = \int_{\mathbb{R}} J(u)\mathscr{W}_0(u)du\,.$$

Macroscopic scale

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Let $J = (J_1, J_1)$ be a pair of functions independent of k then

$$\langle \mathscr{W}^\varepsilon(t), J\rangle = \varepsilon \sum_{x\in\mathbb{Z}} \mathbb{E}_{\mu^\varepsilon}\left[e\left(t\varepsilon^{-1}, x\right)\right] J_1(\varepsilon x) + \mathcal{O}_J(\varepsilon)\,.$$

To understand the behavior of the energy, we have to understand the one of $\mathscr{W}^\varepsilon$.

Historical results of BJKO

BOS [ARMA'10] and SSS [CMP'19] proved that $\mathscr{W}^{\varepsilon}$ converges to f where

$$\partial_t f(t,u,k,i) - \frac{\mathsf{v}_B(k)}{2\pi}\partial_u f(t,u,k,i) = \mathscr{L}_B[f](t,u,k,i)\,.$$

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Here

$$\mathscr{L}_B f(t,u,k,i) = \sum_{j=1}^{2}\int_{\mathbb{T}} \theta_{i,B}^2(k)R(k,k')\theta_{j,B}^2(k')\,\left(f(t,u,k',j) - f(t,u,k,i)\right)dk'\,.$$

$$\mathbf{v}_B(k) = \frac{\sin(\pi k)\cos(\pi k)}{\sqrt{\sin^2(\pi k) + \frac{B^2}{4}}}$$

and

$$\theta_{1/2,B}^2 = \frac{1}{2} \pm \frac{B}{4\sqrt{\sin^2(\pi k) + \frac{B^2}{4}}}\,.$$

Introduction to Random Walks

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ρ is solution of

$$\partial_t \rho(t,u) = \frac{1}{2}\Delta[\rho](t,u)\,.$$

Brownian motion induces diffusion.

A jump process

$$\partial_t f(t,u,k,i) - \frac{\mathsf{v}_B(k)}{2\pi}\partial_u f(t,u,k,i) = \mathscr{L}_B[f](t,u,k,i)\,.$$

Mesoscopic scale

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- $\big(K(0), I(0)\big) = (k,i)\,.$
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Then

$$f(t,u,k,i) = \mathbb{E}_{(k,i)}\Big[f^0\big(Z_u(t), K(t), I(t)\big)\Big] \longrightarrow \rho(t,u) = \mathbb{E}\Big[\rho_0\big(\mathscr{B}_u(t)\big)\Big]\,.$$

Processus de Lévy stables

Soient $\alpha \in (1,2)$ et σ une mesure sur $\mathbb{R}^*$ telle que $d\sigma(r) = |r|^{-\alpha-1}dr$.

Alors :

$$\int_{\mathbb{R}^*} \min(1,r^2)d\sigma(r) < +\infty \text{ et } \int_{\mathbb{R}^*} r^2 d\sigma(r) = +\infty\,.$$

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Alors :

$$\partial_t \rho(t,u) = -(-\Delta)^{\frac{\alpha}{2}}[\rho](t,u) \longrightarrow \partial_t \rho(t,u) = \Delta[\rho](t,u).$$

Study of a Random Walk

Let $\mathcal{N}(t)$ be the number of jumps until time t.

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$$\frac{1}{N} Z_{Nu}(N^{\alpha_B} t) = u + \frac{1}{N}\sum_{n=0}^{\lfloor N^{\alpha_B} t\rfloor} \lambda_B\left(K_n, I_n\right) \mathbf{v}_B\left(K_n\right) \longrightarrow Y_u(\,\cdot\,) \text{ a Lévy process}$$

Study of a Random Walk

Let $\mathcal{N}(t)$ be the number of jumps until time t.

$$Z_u(t) = u + \frac{1}{2\pi}\int_0^t \mathbf{v}_B\left(K(s)\right) ds \ = u + \sum_{n=0}^{\mathcal{N}(t)} \lambda_B\left(K_n, I_n\right)\mathbf{v}_B\left(K_n\right). \longrightarrow \mathcal{B}_u^N(t) = u + \frac{1}{N}\sum_{n=0}^{\lfloor N^2 t\rfloor} X_n$$

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If we define

$$\rho(t,u) = \mathbb{E}\left[\rho_0\left(Y_u(t)\right)\right] \longrightarrow \rho(t,u) = \mathbb{E}\left[\rho_0\left(\mathcal{B}_u(t)\right)\right].$$

Then

$$\partial_t \rho(t,u) = -(-\Delta)^{\frac{\alpha_B}{2}}[\rho](t,u) \longrightarrow \partial_t \rho(t,u) = \Delta[\rho](t,u).$$

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Cane [preprint] : What happens if we replace B by $B_N = BN^{-\delta}$?

An interpolation process

Let $B_N = BN^{-\delta}$ with $\delta > 0$. Now we work with the array (K_n^N, I_n^N).

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$$d\nu_\delta(r) \begin{cases} |r|^{-\frac{3}{2}-1} dr \text{ if } \delta > \frac{1}{2} \\ h_B(r) dr \text{ if } \delta = \frac{1}{2} \\ |r|^{-\frac{5}{3}-1} dr \text{ if } \delta < \frac{1}{2} \end{cases}$$

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$$\lim_{N\to\infty} N^{\alpha_\delta}\pi_{B_N}\left(\left\{(k,i), \lambda_{B_N}(k,i)\mathbf{v}_{B_N}(k) > Nr\right\}\right) = \nu_\delta(r, +\infty).$$

With

$$\alpha_\delta = \frac{5-\delta}{3} \text{ if } \delta < \frac{1}{2} \quad \text{and} \quad \alpha_\delta = \frac{3}{2} \text{ if } \delta \geq \frac{1}{2}.$$

Theorem **[Cane preprint]** : $N^{-1}Z_{Nu}^N(N^{\alpha_\delta}\cdot)$ converges to $Y_u^\delta(\cdot)$.

An interpolation P.D.E

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Theorem [Cane, preprint] :

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Transition graph

$$\partial_t f^N(t,u,k,i) - \mathbf{v}_{B_N}(k)\partial_u f^N(t,u,k,i) = \mathscr{L}_{B_N}[f](t,u,k,i)\,.$$

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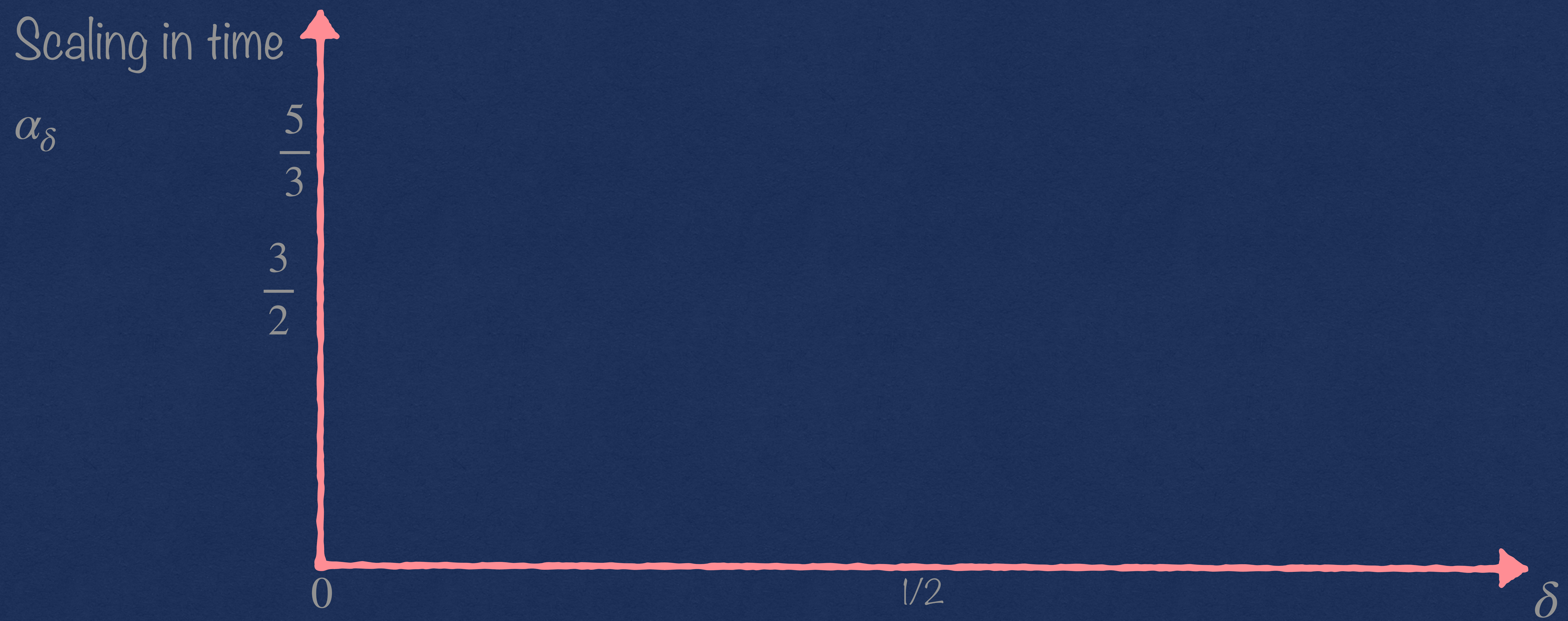

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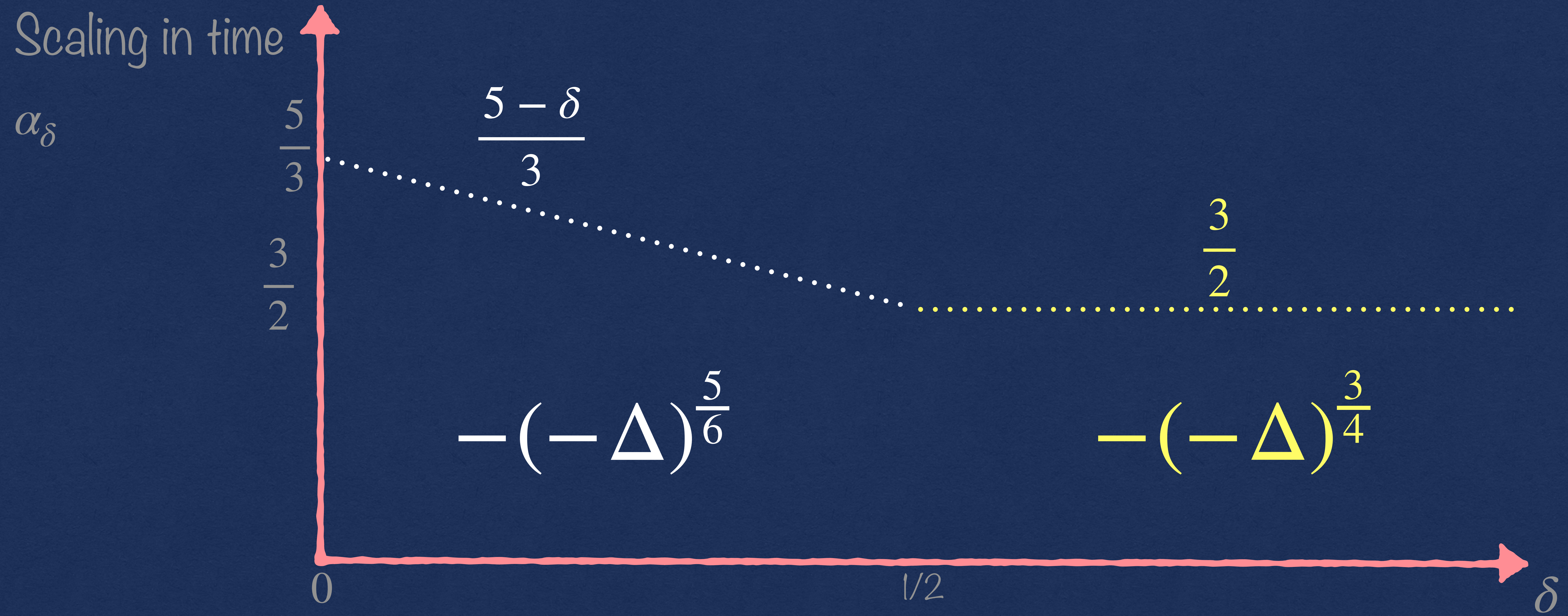

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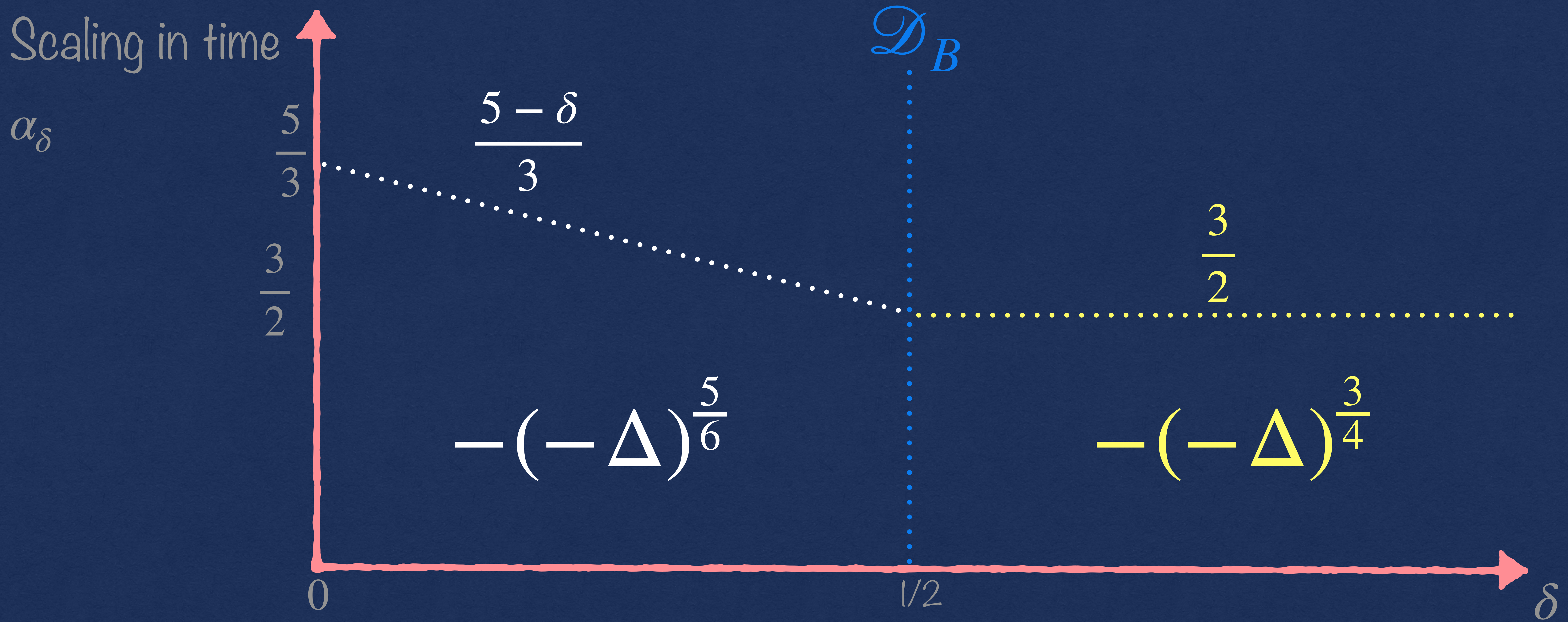

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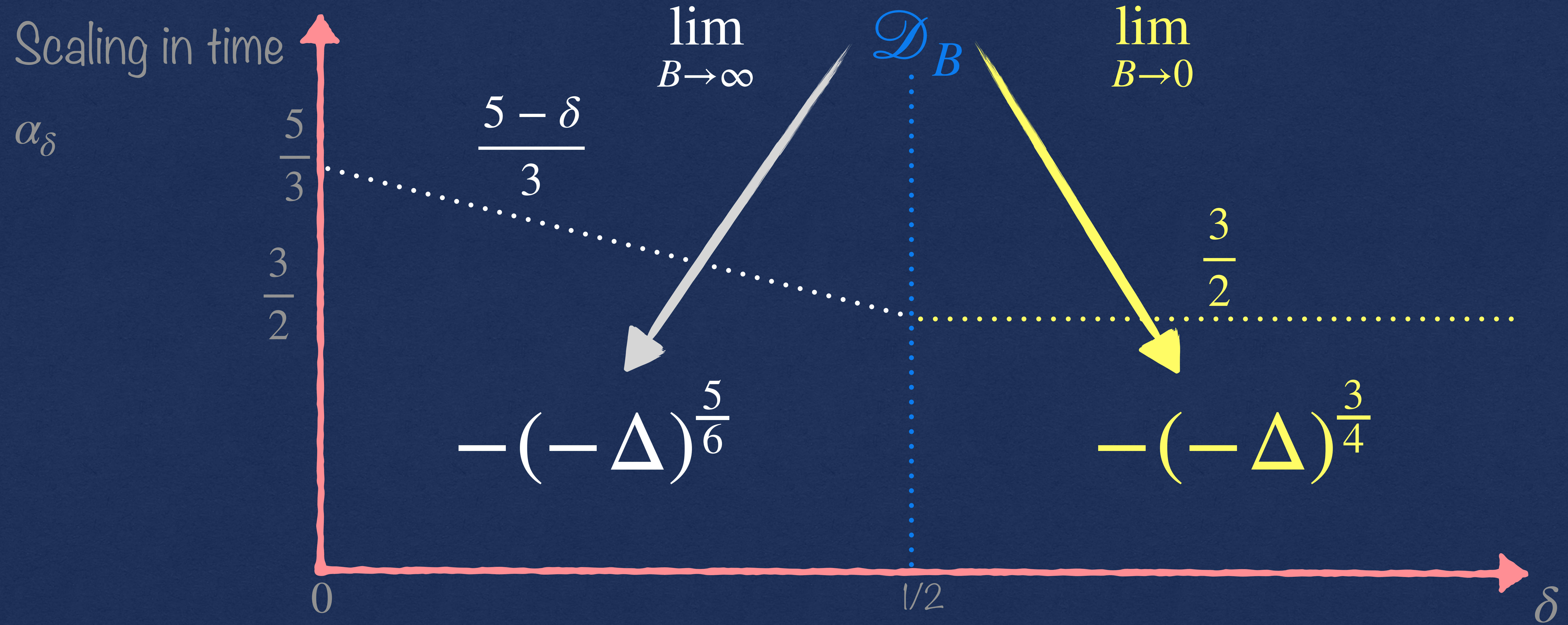

THANK YOU FOR YOUR ATTENTION