Localization effects due to a random magnetic field on heat transport in a harmonic chain

Gaëtan Cane
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03.18.21

Part I

Fourier's law and physical motivations

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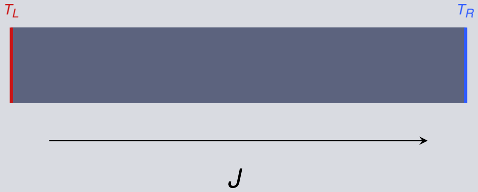
1822 : **Fourier's *experimental* law**:



Fourier’s law

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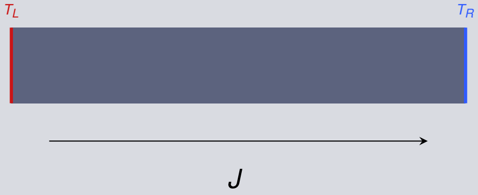

$$J(x) = -K\nabla T(x),$$

where K is the heat conductivity of the system.

Harmonic chain in one dimension

We study a system of N particles.

Harmonic chain in one dimension

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We study a system of N particles.

We denote the position of the particle i by x_i, its speed by $\dot{x}_i$ and its acceleration by $\ddot{x}_i$, then we have for each $i \in \{1, \cdots, N\}$:

$$m_i \ddot{x}_i = (x_{i+1} + x_{i-1} - 2x_i),$$

with $x_0 = x_{N+1} = 0$.

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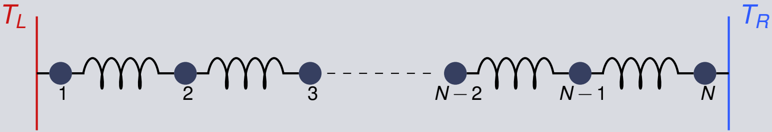
Part II

Historical results on the harmonic chain

Introduction of heat bath in harmonic chain

We attache each end of the chain to a heat bath at different temperature.



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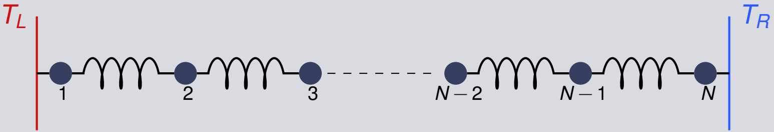

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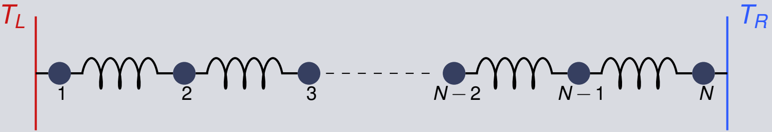

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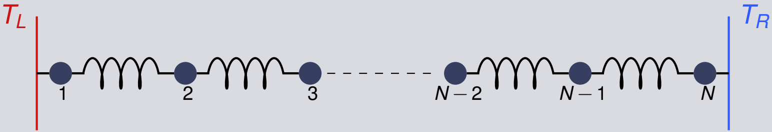

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Stationnary state

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In order to observe the heat current we need to wait until the system reaches the stationnary state. Fourier's law is then written as:

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What happens if we put some disorder on the chain ?

Introduction of random masses

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In 1971,Rubin and Greer worked on the same system as Casher and Lebowitz but with **free boundary conditions** and get:

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Part III

Introduction of a random magnetic field

Presentation of the system

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Work in collaboration with *Junaid Majeed Bhat*, *Abhishek Dhar* and *Cédric Bernardin*.

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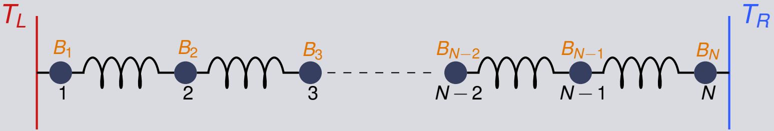

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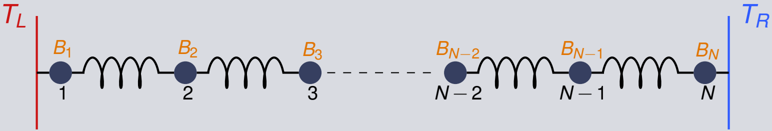

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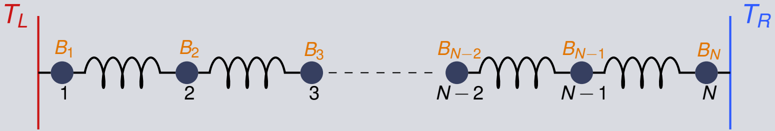

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A particle labelled i is now a two dimensional vector (x_i, y_i). Let $i \in \{1, \cdots, N\}$ then:

$$\ddot{x}_i = (x_{i+1} + x_{i-1} - 2x_i) + (\delta_{i,1} T_L + \delta_{i,N} T_R)\sqrt{2}\mathcal{W}_i^x - (\delta_{i,1} + \delta_{i,N})\dot{x}_i$$

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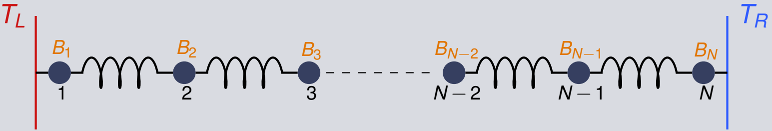

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By using Fourier transform in time we get for all $\omega \in \mathbb{R}$:

$$\begin{pmatrix} 2+1-\omega^2 & -1 & 0 & -B_1 & 0 & 0 \\ -1 & 2-\omega^2 & -1 & 0 & -B_2 & 0 \\ 0 & -1 & 2+1-\omega^2 & 0 & 0 & -B_3 \\ B_1 & 0 & & 2+1-\omega^2 & -1 & 0 \\ 0 & B_2 & 0 & -1 & 2-\omega^2 & -1 \\ 0 & 0 & B_3 & 0 & -1 & 2+1-\omega^2 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{y}_1 \\ \tilde{y}_2 \\ \tilde{y}_3 \end{pmatrix} = \begin{pmatrix} \tilde{\mathcal{W}}_1^x \\ \tilde{\mathcal{W}}_2^x \\ \tilde{\mathcal{W}}_3^x \\ \tilde{\mathcal{W}}_1^y \\ \tilde{\mathcal{W}}_2^y \\ \tilde{\mathcal{W}}_3^y \end{pmatrix}.$$

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In general case we can write:

$$\begin{pmatrix} \Pi(\omega) & \mathcal{B}(\omega) \\ -\mathcal{B}(\omega) & \Pi(\omega) \end{pmatrix} \begin{pmatrix} X(\omega) \\ Y(\omega) \end{pmatrix} = \begin{pmatrix} \mathcal{W}^x(\omega) \\ \mathcal{W}^y(\omega) \end{pmatrix}.$$

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where:

$$G_1(\omega) = \frac{1}{\Pi(\omega) + \mathcal{B}(\omega)[\Pi(\omega)]^{-1}\mathcal{B}(\omega)} \quad \text{and} \quad G_2(\omega) = -G_1(\omega)\mathcal{B}(\omega)[\Pi(\omega)]^{-1}.$$

Heat current and Net transmission

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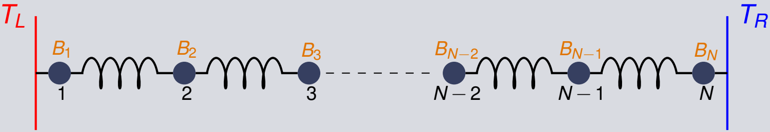

If $\overrightarrow{F_L}$ is the force on the 1st oscillator due to the left reservoir then:

$$\overrightarrow{J} = \overrightarrow{F_L} \cdot (x_1, y_1)^T.$$

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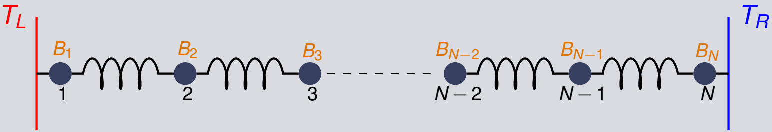

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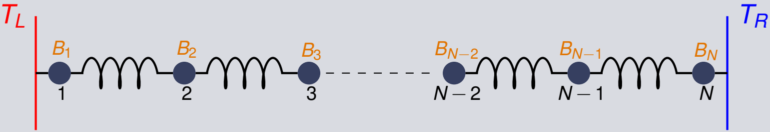

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After computations we get:

$$\langle J \rangle = (T_L - T_R) \int_{-\infty}^{+\infty} \omega^2 \mathcal{T}(\omega) d\omega,$$

with:

$$\mathcal{T}(\omega) = \frac{4}{\pi} \left[|G_1(\omega)_{1,N}|^2 + |G_2(\omega)_{1,N}|^2 \right].$$

Plot of the Net transmission

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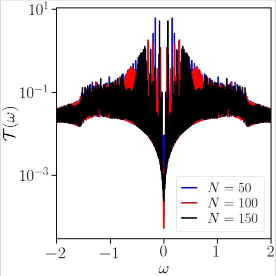
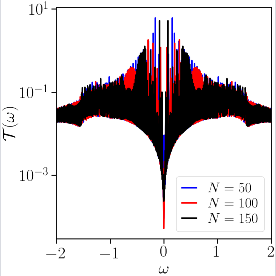


Net transmission for a constant magnetic field.

Plot of the Net transmission



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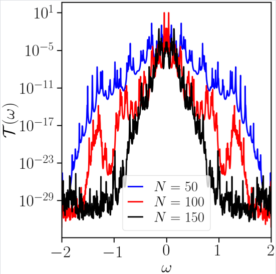

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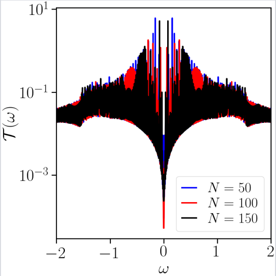

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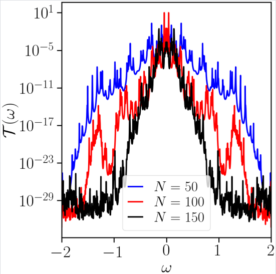

Net transmission for a random magnetic field.

- Randomness causes supression of the net transmission.

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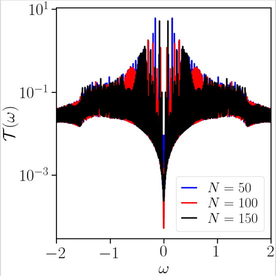

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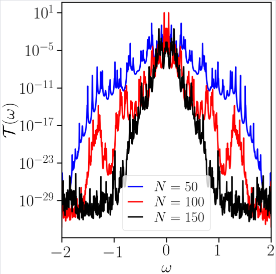

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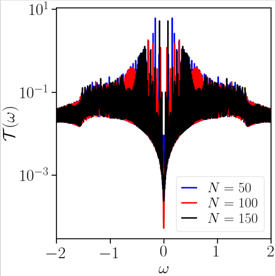

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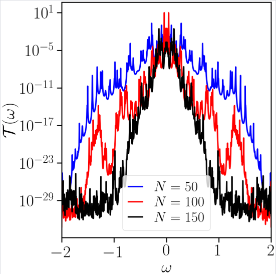

Net transmission for a random magnetic field.

- Randomness causes supression of the net transmission.
- $\mathcal{T}$ is a decreasing function in N.
- $\mathcal{T}$ is higher near $\omega = 0$.
- Normal modes of energy ω are localized with localization length λ.

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We recall that:

$$\langle J \rangle = (T_L - T_R)\frac{4}{\pi}\int_0^\infty \left[|G_1(\omega)_{1,N}|^2 + |G_2(\omega)_{1,N}|^2 \right].$$

Let $\omega \in \mathbb{R}$. We define the following matrice:

$$\mathcal{G}_{1,N}(\omega) = \begin{pmatrix} G_1(\omega)_{1,N} & G_2(\omega)_{1,N} \\ -G_2(\omega)_{1,N} & G_1(\omega)_{1,N} \end{pmatrix}.$$

Green function as product of random matrices

Gaëtan Cane
LJAD, UCA

We recall that:

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After a change of variable we can write that:

$$\begin{aligned} \mathcal{G}_{1,N}^{-1}(\omega) &= \Omega_L \prod_{i=1}^{N} \begin{pmatrix} (2-\omega^2)I_2 - \mathbf{i}\omega B_i J & I_2 \\ -I_2 & 0 \end{pmatrix} \Omega_R \\ &= \Omega_L \prod_{i=1}^{N} \Omega_i(\omega) \Omega_R \end{aligned}$$

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Furstenberg's theorem gives us :

$$\lim_{N\to\infty} \frac{1}{N} \log \left(\left\| \prod_{i=1}^{N} \Omega_i(\omega) \right\| \right) = \lambda(\omega),$$

where λ is the Lyapunov exponent.

Formal expression for the size of the current

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Hence, for N large enough, a formal reasoning leads to:

$$\left\|\prod_{i=1}^{N} \Omega_i(\omega)\right\| \sim \exp(N\lambda(\omega)).$$

$$\langle J \rangle \quad = \quad \frac{8(T_L - T_R)}{\pi} \int_0^{+\infty} \omega^2 \left[|G_1(\omega)_{1,N}|^2 + |G_2(\omega)_{1,N}|^2 \right] d\omega$$

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Boundary conditions and **Lyapunov exponent** give us the size of the current.

Reduction to a product of 2×2 random matrices

Gaëtan Cane
LJAD, UCA

We define:

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{i} & 1 \\ -\mathbf{i} & 1 \end{pmatrix}.$$

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Then we can prove that:

$$\prod_{i=1}^{N} \Omega_i(\omega) = \begin{pmatrix} U^\dagger & 0 \\ 0 & U^\dagger \end{pmatrix} \prod_{i=1}^{N} \begin{pmatrix} (2-\omega^2) I_2 + \omega B_i \sigma^z & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix},$$

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with:

$$\forall i \in \{1, \cdots, N\}, \quad f_{i+1}^- = (2 - \omega^2 - \omega B_i) f_i^- - f_{i-1}^-.$$

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From the definition of $(f_i^-)_i$ we have:

$$\begin{pmatrix} f_N^- \\ f_{N-1}^- \end{pmatrix} = \prod_{i=1}^{N} \begin{pmatrix} 2-\omega^2-B_i\omega & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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From a Markov chain to a Markov process

From the definition of $(f_i^-)_i$ we have:

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$$f_{i+1}^- - 2f_i^- + f_{i-1}^- = -(\omega^2 + \omega B_i) f_i^- .$$

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Continuum limit leads to:

$$\ddot{f}(t) = -(\omega^2 + \omega B(t)) f(t).$$

We define $\eta_t = B(t) - \langle B \rangle$. Hence, we get:

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This is an equation of the form:

$$\dot{z}(t) = A_0 z(t) + \omega\eta_t A_1 z(t).$$

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Theorem (Wihstutz in 1999)

Let $c \in \mathbb{R}$. For a stochastic differential equation of the form,

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with:

$$A_0 = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix} \quad A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Lyapunov exponent for particular class of SDE

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Then we have:

- *If $c > 0$ then $\lambda(\varepsilon) = \frac{\varepsilon^2}{8c} + \mathcal{O}(\varepsilon^6)$.*
- *If $c < 0$ then $\lambda(\varepsilon) = \sqrt{-c} + \mathcal{O}(\varepsilon^2)$.*
- *If $c = 0$ then $\lambda(\varepsilon) = \alpha_0 \varepsilon^{2/3} + \mathcal{O}(\varepsilon)$.*

where λ is the Lyapunov exponent associated to the process $(z_t)_t$.

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$$\tilde{t} = \sqrt{\omega}t.$$

Lyapunov exponent of the Markov process

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When $\langle B\rangle \neq 0$ the following change of time is used:

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Using again the previous Theorem and the fact that $\lambda = \tilde{\lambda}\sqrt{\omega}$ we get:

- For $\langle B\rangle > 0$, $\lambda(\omega) = \frac{\omega}{8\langle B\rangle} + O(\omega^{5/4})$.
- For $\langle B\rangle < 0$, $\lambda(\omega) = \sqrt{-\langle B\rangle}\omega^{1/2} + O(\omega^{5/2})$.

Plot of the Lyapunov exponent

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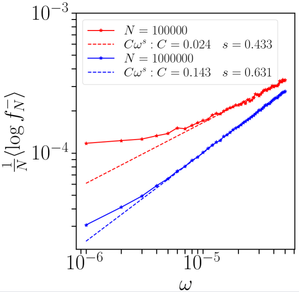

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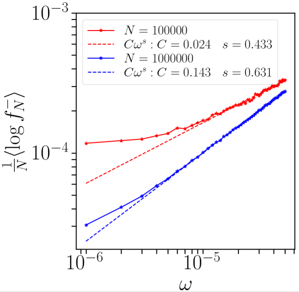

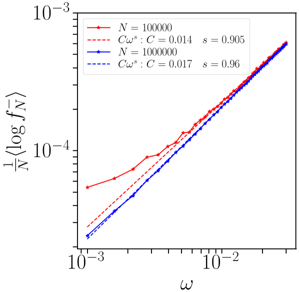

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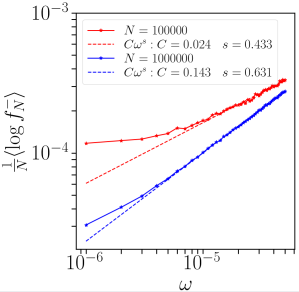

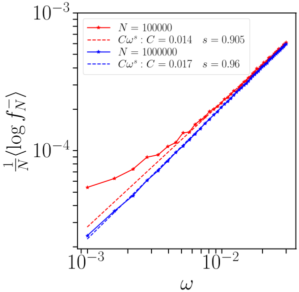

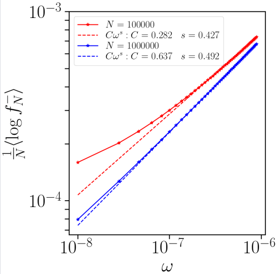

Back to the heat current

Gaëtan Cane
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We recall that:

$$\langle J \rangle \sim (T_L - T_R) \int_0^\infty b_c(\omega)\omega^2 \exp\left(-N\lambda(\omega)\right) d\omega.$$

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In our model we have $b_c(\omega) \sim 1$.

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In our model we have $b_c(\omega) \sim 1$. Hence, we get:

- If $\langle B \rangle \neq 0$ then $\langle J \rangle \sim N^{-5/2}$.
- If $\langle B \rangle = 0$ then $\langle J \rangle \sim N^{-9/2}$.

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